APR 13 1928

PERIODICAL ROOM
GENERAL LIBRARY
UNIV. OF MICH.

AMERICAN

JOURNAL OF MATHEMATICS

FOUNDED BY THE JOHNS HOPKINS UNIVERSITY

EDITED BY

E. W. CHITTENDEN
UNIVERSITY OF IOWA

A. B. COBLE
THE JOHNS HOPKINS UNIVERSITY

ABRAHAM COHEN
THE JOHNS HOPKINS UNIVERSITY

G. C. EVANS
RICE INSTITUTE

FRANK MORLEY, CHAIRMAN
THE JOHNS HOPKINS UNIVERSITY

WITH THE COÖPERATION OF

E. T. BELL

F. D. MURNAGHAN

W. A. MANNING

PUBLISHED UNDER THE JOINT AUSPICES OF

THE JOHNS HOPKINS UNIVERSITY
AND
THE AMERICAN MATHEMATICAL SOCIETY

Volume L, Number 2

APRIL, 1928

THE JOHNS HOPKINS PRESS
BALTIMORE, MARYLAND
U. S. A.

CONTENTS

A Theorem on Correspondences on Algebraic Curves. By S. LEFSCHETZ, 159

Concerning the Structure of a Continuous Curve. By GORDON T. WHYBURN, ... 167

Compound Statements on Four Classes. By FRANCES E. BAKER, . 195

A Memoir on the Invariants of Biternary Quantics. By O. E. GLENN, 209

Transitive Substitution Groups Containing Regular Subgroups of Lower Degree. By F. E. JOHNSTON, ... 231

Generalized Green's Functions for Compatible Differential Systems. By W. W. ELLIOTT, ... 243

An Expansion Involving p Inseparable Parameters Associated with a Partial Differential Equation. By CHESTER C. CAMP, - 259

On the Asymptotic Evaluation of Functions Defined by Contour Integrals. By D. M. WRINCH, ... 269

Generalized Quaternion Algebras and the Theory of Numbers. By LOIS W. GRIFFITHS, ... 303

THE AMERICAN JOURNAL OF MATHEMATICS will appear four times yearly.

The subscription price of the JOURNAL for the current volume is $6.00 (foreign postage 25 cents); single numbers $1.50.

A few complete sets of the JOURNAL remain on sale.

Papers intended for publication in the JOURNAL may be sent to any of the Editors.

Editorial communications may be sent to Dr. A. COHEN at the Johns Hopkins University.

Subscriptions to the JOURNAL and all business communications should be sent to THE JOHNS HOPKINS PRESS, BALTIMORE, MARYLAND, U. S. A.

Entered as second-class matter at the Baltimore, Maryland, Postoffice, acceptance for mailing at special rate of postage provided for in Section 1103, Act of October 3, 1917, Authorized on July 3, 1918

PRINTED BY THE J. H. FURST COMPANY
BALTIMORE, MD.

o

s

a
t
T
h
(
t
l

A Theorem on Correspondences on Algebraic Curves.*

By S. Lefschetz.

1. The object of the present paper is to prove the following

Theorem. *The Abelian functions attached to a general algebraic curve of genus p, C_p, are without complex multiplication.*

The equivalent geometric property is: *C_p carries only valence correspondences.*

This basic proposition was first stated by Hurwitz † and several proofs are found in the literature, ‡ but so far as we know, they are all open to objection if only on the ground of being essentially too complicated. Two proofs are given here, one purely function theoretic, the other geometric,§ both very straightforward. The method followed consists in showing that: (a) If the Abelian functions of every C_p have a complex multiplication, then there exists a fixed complex multiplication common to them all, i. e. independent of C_p. (b) There exist C_p's without complex multiplication.

2. Let $\omega = \| \omega_{j\mu} \|$, $(j = 1, 2, \cdots p; \mu = 1, 2, \cdots, 2p)$ be the period matrix of a set of p independent integrals of the first kind $u_1, \cdots, u_p$ attached to C. As Riemann has shown there exists an associated alternating matrix $\boldsymbol{c}$ of order $2p$, with rational terms, such that: ‖

I. $\omega c \omega' = 0$.

II. $\omega c \bar{\omega}'$ is a definite Hermitian matrix.

---

* For a résumé of their theory and complete references see the forthcoming Report of the National Research Council: *Selected Topics in Algebraic Geometry*, Chs. 7, 16.

† *Mathematische Annalen*, Vol. 28 (1887), pp. 561-585.

‡ See Severi, *Mathematische Annalen*, Vol. 74 (1913), pp. 515-544; Enriques-Chisini, *Teoria Geometrica delle Equazioni*, Vol. 3, p. 493.

§ This geometric proof is based on a theorem of Severi's recently completed by Zariski, *American Journal of Mathematics*, Vol. 50 (1928). At the end of his paper Zariski gives another proof of the theorem here considered similar to ours. His two proofs and ours have been obtained independently. In a Note which just appeared in the *Rendiconti dei Lincei*, Ser. 6, Vol. 6 (1927), p. 435, Severi returned to his theorem and completed it.

‖ We use substantially the matrix notations of Bôcher's *Higher Algebra*. The matrix $\| a_{ij} \|$ is designated by $\boldsymbol{a}$, its transverse by $\boldsymbol{a}'$, the matrix of the conjugates of its terms by $\bar{\boldsymbol{a}}$.

These conditions characterize ω as a *Riemann matrix* in the sense of Scorza.*

Now any algebraic correspondence T on C results in a relation,

$$\alpha\omega = \omega a \tag{1}$$

(α square of order p, a square of order $2p$ and with terms rational) defining a multiplication of the Abelian functions attached to ω (more briefly: multiplication of ω). As shown by Scorza (1) leads to a relation

$$\omega d \omega' = 0 \tag{2}$$

analogous to I except that d need not be alternating.

Explicitly (2) implies that

$$\sum_{\mu,\nu} d_{\mu\nu}\omega_{j\mu}\omega_{k\nu} = 0, \quad (j, k = 1, 2, \cdots, p). \tag{3}$$

From this follows that if u, v are arbitrary integrals of the first kind with Ω_μ, Ω_ν' for μth periods, then

$$\sum d_{\mu\nu}\Omega_\mu\Omega_\nu' = 0 \tag{4}$$

Conversely of course if (4) holds whatever u, v so does (3) and also (2), so that (2) and (4) are equivalent.

When c and d are linearly dependent, the multiplication (1) is *ordinary* and T is a *valence* correspondence, otherwise the multiplication is *complex* and T is a *singular* correspondence. Thus valence correspondences are characterized by the fact that they impose no new relation on ω other than I.

The question arises however if the mysterious relations existing between the periods for $p \geqq 4$ † imply a complex multiplication, or a singular T. The answer according to our theorem is negative.

3. We recall that if we choose a set of retrosections as the fundamental circuits on C_p, and the related normal integrals as the independent set of integrals, ω assumes the form

$$\| \delta_{jk}; \tau_{jk} \|, \quad (j, k = 1, 2, \cdots p)$$

where $\mathfrak{d}$ is the unit matrix of order p. In place of I and II we have now

I′. τ is symmetrical.

---

* A summary of his work and application to correspondences on a curve with complete references will be found in the Report, Chs. 15, 16, 17.

† When $p = 4$ there is a unique relation discovered by Schottky; beyond that we are completely in the dark.

II′. Let $\tau_{jk} = \tau'_{jk} + i\,\tau''_{jk}$. Then the quadratic form $\sum \tau''_{jk}\, x_j x_k$ is definite.

The periods τ_{jk}, are analytical functions of $3p-3$ parameters for $p > 1$, of *one* for $p = 1$.*

The existence of a complex multiplication implies the existence of relations

$$Q_h(\tau_{11}, \tau_{12}, \cdots) = 0; \quad h = 1, 2, \cdots, p(p-1), \tag{5}$$

where the Q's are quadratic polynomials in the τ's with integer coefficients.

4. Consider the τ's as coördinates of a point in a function space S_n, $n = p(p+1)/2$. In that space there exists a certain analytical q-spread M_q, $q \leqq 3p-3$ representing the ∞^q sets of functionally independent normal periods of a generic C_p. For $p = 1$ we have $q = 1$.

Since the coefficients of the Q's in (4) are integers the set of all such systems is denumerable. Let $\{W_i\}$, $i = 1, 2, \cdots$ be the infinite set of the corresponding spreads in some order. If our theorem is false some W_i passes through every point of M_q. In that case I say that *at least one W carries M_q itself*. Let us assume that this is false. Let $A(\tau_{jk}{}^0)$ be an arbitrary point of M_q. Its neighborhood E_q on M_q is an analytical element represented by parametric equations

$$\tau_{jk} - \tau_{jk}{}^0 = \phi_{jk}(t_1, \cdots, t_q), \quad |\, t_h \,| < \alpha$$

where the τ's are power series zero at $t_1 = \cdots = t_q = 0$, convergent and with a Jacobian matrix of rank q within the range considered. The intersections of E_q with the W's are a denumerable set $\sum$ of analytical spreads of less than q dimensions. Since the sections of E_q by a set of parallel hyperplanes of S_n constitute a non denumerable set of non intersecting analytical $q-1$ spreads on E_q, one of them at least will be different from every one of those of $\sum$. Hence E_q carries a $q-1$ spread behaving exactly like M_q with respect to the set $\{W_i\}$. Proceeding step by step we come to a situation such as that assumed at the start but with $q = 1$. Assume then $q = 1$. If no W_i carries M_1, or a definite analytical element E_1 of M_1, it intersects it in a finite number of points. Hence the intersections of the W's with E_1 constitute a denumerable set of points on E_1. Every point of E_1 belongs to the set—which is absurd since E_1 is a two dimensional real analytical element and the set of all its points has the power of the continuum. It follows that M_q is carried

* See for example Picard, *Cours d'Analyse*, Vol. 2, 3rd ed., p. 578.

by at least one W. In other words, *either our theorem holds or else there is a fixed complex multiplication for the period matrix* ω *of every* C_p.*

The task that we are facing is then to show that for every p *there is at least one curve of that genus whose* ω *is without complex multiplication or whose correspondences all have valence.*

We shall show that there exists for every p at least two systems of curves having the requisite property, one of them composed of arbitrary hyperelliptic curves.

5. *Proof that arbitrary hyperelliptic curves carry only valance correspondences.* Consider the general hyperelliptic curve C_p:

$$y^2 = \prod_{i=1}^{2p+2} (x - a_i). \tag{6}$$

We have to show that when the a's are arbitrary there is no complex multiplication attached to the period matrix ω. This is true for $p = 1, 2$, since the corresponding period matrices are arbitrary Riemann matrices of their genus. We shall therefore assume $p > 2$.

Let K be the two-sheeted Riemann surface of C_p, L_μ a loop drawn on K from $x = a$ to the branch point a_μ in the first sheet and back to the point $x = a$ in the second. We assume that when we turn around the point in the positive direction the loops follow one another around $x = a$ in the order in which they are numbered.

The system of linear cycles

$$\gamma_{2\mu-1} = L_{2\mu} - L_{2\mu-1}$$
$$\gamma_{2\mu} = -L_{2\mu} + L_{2\mu+1} - L_{2\mu+2} + \cdots + L_{2p+1}$$

constitutes a set of retrosections. An equivalent system is as follows: We draw non-intersecting cuts $a_{2\mu-1}a_{2\mu}$. Then $\gamma_{2\mu-1}$ is deformable into a circuit surrounding the cut just named in the first sheet, $\gamma_{2\mu}$ into a circuit crossing the cuts of order μ and $p+1$ and no others. The circuits can furthermore be so drawn that $\gamma_{2\mu-1}$ and $\gamma_{2\mu}$ cross one another but meet no other γ.†

---

* A highly sophisticated way of looking at the matter is as follows: It can readily be shown that of two analytical elements E_q, E_r, either one carries the other, or else their intersection has relatively to one of them, say E_q zero Lebesgue measure,—the measure being taken with respect to E_q and its $2q$ dimensional real volume elements. Hence our theorem is equivalent to the statement that the W's intersect E_q in a set of measure zero, or, in the usual language of real variables to the following: *Almost all algebraic curves carry only valence correspondences.*

† See for example Appell et Goursat, *Théorie des fonctions algébriques*, p. 124.

6. Let then u, v be two integrals of the first kind attached to C, Ω_μ, Ω_μ', their periods relatively to γ_μ. Since the γ's are retrosections we have

$$\sum (\Omega_{2\mu-1}\Omega'_{2\mu} - \Omega_{2\mu}\Omega'_{2\mu-1}) = 0 \tag{7}$$

and we must show that any bilinear relation with integer coefficients,

$$\sum c_{\mu\nu}\Omega_\mu\Omega'_\nu = 0, \tag{8}$$

satisfied whatever u, v, necessarily reduces to the preceeding. In so doing we may assume $p > 2$, and the theorem proved for $p-1$, then proceed by induction.

The method to be followed will consist in making $a_{2\mu-1} \to a_{2\mu}$, $\mu \leqq p$, and finding the limits of the Ω's for u, v suitably chosen. From (8) we shall thus deduce a limiting relation for a curve of genus $p-1$ from which our theorem will readily follow.

7. It will be convenient to write $a_{2\mu-1} = z$, $a_{2\mu} = z'$. We choose

$$u = \int [P(x)(x-z')/y]\,dx, \quad v = \int [Q(x)(x-z')/y]\,dx,$$

P and Q linearly independent polynomials of degree $p-2$. We are going to assume that $z' \to z$ in such manner that $z - z'$ remains real and positive. In other words z' approaches z along a segment parallel to the real axis and to the right of the point.

8. Let us introduce the auxiliary C_{p-1}:

$$y_1^2 = (x-a_1)\cdots(x-a_{2\mu-2})(x-a_{2\mu+1})\cdots(x-a_{2p+2}).$$

We have in

$$u_1 = \int (P/y_1)\,dx, \quad v_1 = \int (Q/y_1)\,dx$$

two perfectly arbitrary integrals of the first kind attached to C_{p-1}.

Let K_1 be the Riemann surface of C_{p-1} derived from K by suppressing the branch points $a_{2\mu-1}$, $a_{2\mu}$.

The circuits γ_ρ, $\rho \neq 2\mu-1$ or 2μ, constitute a set of restrosections for K_1. Let $\Omega_{1\rho}$, $\Omega'_{1\rho}$ be the corresponding periods of u_1, v_1. When $z' \to z_1$ it is at once seen that $\Omega_\rho \to \Omega_{1\rho}$, $\Omega_\rho' \to \Omega'_{1\rho}$, $\rho \neq 2\mu-1$ or 2μ. Furthermore $\Omega_{2\mu-1}$ and $\Omega'_{2\mu-1}$ both $\to 0$ for in the limit they are equal to u_1, v_1, integrated along a circuit surrounding an ordinary point of K_1.

9. There remains then merely to find the limits of $\Omega_{2\rho}$, $\Omega'_{2\mu}$. Let us confine z' to a circle of center z and radius ϵ as small as we please and in any case such that all branch points other than z and z' are exterior to it. We have then

$$\Omega_{2\mu} = 2\int_{z'}^{z+\epsilon} du + 2\int_{z+\epsilon}^{a_{2p+1}} du = 2I_1 + 2I_2.$$

Here I_1 and I_2 designate the two definite integrals, the paths of integration being a straight line for I_1, some finite regular arc exterior to the above circle for I_2. The latter is a holomorphic function of z' about the point z therefore

$$\lim_{z' \to z} \int_{z+\epsilon}^{a_{2p+1}} du = \int_{z+\epsilon}^{a_{2p+1}} du_1.$$

Within and on a certain circle $|x - z| = \eta$, where $\eta > \epsilon$ and is to be fixed in the sequel, $P(x)/y_1$ is holomorphic hence within that same circle, $|P(x)/y_1| < M$, a fixed number, and

$$\lim_{z' \to z} \int_{z+\epsilon}^{a_{2p+1}} du = \int_{z}^{a_{2p+1}} du_1 + \zeta, \ |\zeta| < M$$

the path of integration being some finite regular arc joining the end points and avoiding the critical points. Also $\xi = x - z$ is real on the path of integration of I_1 with $0 < z' - z \leqq \xi \leqq \epsilon$. Hence on that path

$$[(x - z')/(x - z)]^{\frac{1}{2}} = [1 - (z' - z)/\xi]^{\frac{1}{2}} \text{ is real and } \leqq 1.$$

Therefore on the path of integration the integrand of I_1 is in absolute value $< M$, and $|I_1| < M(z + \epsilon - z') < 2M\epsilon$.
It follows

$$\left| \lim_{z' \to z} \Omega_{2\mu} - 2\int_{z}^{a_{2p+1}} du_1 \right| < 6M\epsilon$$

and since ϵ is arbitrarily small $\Omega_{2\mu} \to 2\int_{z}^{a_{2p+1}} du_1$.

We can set by definition $u_1 = \int_{a_{2p+1}}^{\infty} du_1$. Then $\Omega_{2\mu} \to -2u_1(z)$.

With a similar situation for v_1 we shall have $\Omega'_{2\mu} \to -2v_1(z)$.

10. The relation (8) will then yield in the limit a relation

$$Au_1(z)v_1(z) + Bu_1(z) + Cv_1(z) + \sum c_{\rho\sigma}\Omega_{1\rho}\Omega'_{1\sigma} = 0 \tag{9}$$

where A, B, C are independent of z and ρ, σ run from 1 to $2p$ omitting $2\mu - 1$

and 2μ. If z follows the closed circuit γ_ν, $\nu \neq 2\mu - 1$, 2μ, the increment of the left side must be zero. Hence

$$A(\Omega_{1\nu} v_1(z) + \Omega'_{1\nu} u_1(z)) = \text{const.}$$

Since the periods Ω_1 are not all zero, and since u_1, v_1 are like P, Q, linearly independent, it follows that $A = 0$. For the same reason then $B = C = 0$. Therefore (9) reduces to

$$\sum c_{\rho\sigma} \Omega_{1\rho} \Omega'_{1\sigma} = 0; \quad \rho, \sigma \neq 2\mu - 1 \text{ or } 2\mu.$$

But C_{p-1} is an arbitrary hyperelliptic curve of genus $p - 1 \geqq 2$, and u_1, v_1, are arbitrary integrals of the first kind attached to it. Our theorem being assumed true for $p - 1$ it follows that

$$c_{2\nu-1,\, 2\nu} = -c_{2\nu,\, 2\nu-1} = c, \text{ a fixed integer, } \nu \neq \mu$$

while all other coefficients $c_{\rho\sigma}$ are zero, $\rho, \sigma \neq 2p - 1$ or 2μ.

Since $p > 2$ there are at least two positive integers $\rho, \sigma \leqq p$ and $\neq \mu$. Bring into coincidence $a_{2\rho-1}$ and $a_{2\rho}$ but no other critical points. This time we shall find

$$c_{2\mu-1,2\mu} = -c_{2\mu,2\mu-1} = c$$
$$c_{2\mu-1,\sigma} = c_{\sigma,2\mu-1} = c_{2\mu,\sigma} = c_{\sigma,2\mu} = 0.$$

Since σ is any integer $\leqq p$ and $\neq \mu$, it follows ultimately that $c_{2\mu-1,\, 2\mu} = -c_{2\mu,\, 2\mu-1} = c$, whatever $\mu \leqq p$, all other coefficients $c_{\rho\sigma}$ being zero. In other words (8) merely reduces to (7) which proves our theorem.

11. *Another type of curves carrying only valence correspondences.* Its existence is established by means of the following proposition established by Zariski in the paper already quoted: *On the general curve C of a simple linear system* drawn on a regular algebraic surface† all correspondences have a valence.*

Let x, y, z be homogeneous plane coordinates and consider the linear system $|C|$ of all curves

$$(10) \qquad \sum_{q=0}^{\mu} \sum_{s=0}^{\nu} a_{qs} x^q y^s z^{m-q-s} = 0, \qquad (\mu + \nu = m).$$

* A linear system $|C|$ of curves on the surface F is *simple* whenever the curves C that pass through an arbitrary point A of F are not constrained to pass through other points of F variable with A.

† The surface F is *regular* whenever its linear connection index is zero or, equivalent condition, when it possesses no simple Picard integrals of the first kind. If every circuit on F can be deformed into a point it is certainly regular. It follows that the plane is regular—which is all that we need here.

We shall show that μ, ν can be so chosen that the generic C be of genus p, and that the system be simple. By Zariski's theorem it will then have the requisite property.

12. The generic curve C has just two multiple points with distinct tangents at (1, 0, 0) and (0, 1, 0) with respective multiplicities ν, μ. Its genus is therefore

$$p = \binom{m-1}{2} - \binom{\mu}{2} - \binom{\nu}{2} = (\mu - 1)(\nu - 1)$$

It is evident that given p we can always find μ, ν and therefore a $| C |$ whose curves are of genus p.*

The *dimension* of $| C |$ is one less than the number of distinct coefficients a_{qs} in (10), or $r = (\mu + 1)(\nu + 1) - 1$. The *degree* of $| C |$, or number of variable intersections of two arbitrary C's is

$$d = (\mu + \nu)^2 - \mu^2 - \nu^2 = 2\mu\nu.$$

I say that $| C |$ is simple. For in the contrary case every C through an arbitrary point A of the plane would be constrained to pass through $k - 1 > 0$ other points variable with A. But there are at least two C's going through $r - 1$ arbitrary points. Therefore any two C's would intersect in no less than $(r - 1)k$ variable points and $d \geqq (r - 1)k$. This gives

$$2\mu\nu \geqq k(\mu + 1)(\nu + 1) - 2k$$
$$(k - 2)\mu\nu + k(\mu + \nu - 1) \leqq 0$$

which is manifestly impossible since $k \geqq 2$, $\mu \geqq 1$, $\nu \geqq 1$. Thus $| C |$ is simple and a generic C carries only valence correspondences.

13. *Conclusion.* We have reduced our basic theorem to the more special problem is finding some curves of genus p that carry only valence correspondences. We have two types of curves of the required nature. For the first—generic hyperelliptic curves—the proper behavior has been established by a strictly transcendental method. For the second the same result has been achieved by a geometrical procedure based in part on a theorem due to Zariski. We have therefore given two essentially distinct proofs of the theorem.

PRINCETON UNIVERSITY.

* Snyder has established the existence of curves of any genus p on a non-degenerate quadric, *Bulletin of the American Mathematical Society*, Vol. 15 (1908-09), pp. 1-4. If the quadric is mapped perspectively on a plane his curves give rise to the linear system here considered.

Concerning the Structure of a Continuous Curve.*

By Gordon T. Whyburn.

1. *Introductory. Notation and Definitions. Cyclic Elements.* A continuous curve M is said to be cyclicly connected † provided every two points of M lie together on some simple closed curve which is a subset of M. A cyclicly connected continuous curve C which is a subset of a continuous curve M is said to be a maximal cyclic curve of M if and only if C is not a proper subset of any other cyclicly connected continuous curve which is also a subset of M. The point P of a continuous curve M is an endpoint of M if and only if no simple continuous arc of M has P as one of its interior points.‡ The point P of a *continuum* M is an endpoint of M provided that if N is any subcontinuum of M containing P, then P is not a limit point of any connected subset of $M - N$.§ In this paper the ordinary notation will be used for the sum, product, and difference of point sets. If a symbol X is used to denote a point set, the symbol $\overline{X}$ will be used to denote the set X plus all the limit points of the set X. It is to be understood that the point sets considered in this paper lie in a Euclidean plane.‖ Also the hypothesis that "M is a continuous curve" is understood to imply that the point set M is bounded.

A subset E of a continuous curve M will be called a cyclic element of M

---

* Presented to the American Mathematical Society December 31, 1926. The results in § 5 were presented to the Society June 18, 1927.

† G. T. Whyburn, "Cyclicly Connected Continuous Curves," *Proceedings of the National Academy of Sciences*, Vol. 13 (1927), pp. 31-38.

‡ This definition is equivalent to the one given by R. L. Wilder in his paper "Concerning Continuous Curves," *Fundamenta Mathematicae*, Vol. 7 (1925), p. 358. For a proof of this fact see my paper "Concerning Continua in the Plane," *Transactions of the American Mathematical Society*, Vol. 29 (1927), No. 2, pp. 369-400, Theorem 12, p. 385.

§ See G. T. Whyburn, *loc. cit.*, p. 382. See also H. M. Gehman, *Transactions of the American Mathematical Society*, Vol. 30 (1928), pp. 63-84.

‖ With the aid of some results established recently by Dr. W. L. Ayres on the cyclic elements of a continuous curve in n dimensions it follows that practically all the theorems in the present paper hold true in n dimensions. I have found that all except numbers 24-27 and 29 so hold. In most cases the arguments as given extend to n dimensions without modification, although in a few instances a slightly different proof is necessary.

provided that E is either (a) a maximal cyclic curve of M, (b) a cut point* of M, or (c) an endpoint of M. The cyclic elements of a continuous curve will be referred to by classes (a), (b), and (c) as here defined. That every continuous curve is the sum of its cyclic elements follows from the results, in particular Theorem 7, of my paper "Cyclicly Connected Continuous Curves," cited above. On account of the frequent references to that paper, it will be referred to hereafter as C. C. The following conclusions are easily deduced from the definition of the cyclic elements of a continuous curve M: (1) no two cyclic elements of M have more than one point in common, (2) the common part of every pair of cyclic elements of M is either vacuous or is itself a cyclic element of class (b), and (3) if N, K, and H respectively denote the sum of all the cyclic elements of classes (a), (b), and (c) respectively, then (I) $N \cdot K$ is countable, and (II) $N \cdot H = K \cdot H = 0$. By an *internal point* of a maximal cyclic curve C of a continuous curve M will be meant a point of C which is not a cut point of M. The notation $I(C)$ will be used to designate the set of all the internal points of maximal cyclic curve C of a continuous curve. It is readily seen from (I) above that the set of all the non-internal points of a maximal cyclic curve must be countable. Also it is easy to see that no two maximal cyclic curves of a continuous curve can have an internal point of either in common without being identical.

By an acyclic continuous curve is meant a continuous curve which contains no simple closed curve. In this paper it will be shown in §§ 2 and 3 that if we regard a continuous curve as being composed of its cyclic elements instead of points, then *every continuous curve* is acyclic with respect to these elements. Many of the characteristic properties of acyclic continuous curves are established for any continuous curve which is regarded as being composed of its cyclic elements. In particular it is shown that every continuum of cyclic elements is a continuous curve, and that every connected set of cyclic elements is arcwise connected in the ordinary sense. In § 4 the notions of the *nodes* and *endelements* of a continuous curve are introduced and characterized in various ways, and some of the familiar propositions concerning the endpoints of acyclic continuous curves are shown to hold for the endelements of any continuous curve. In §§ 5 and 6 some theorems will be established relating to the structure of a continuous curve from the ordinary point of view.

* The point P of a connected set M is a cut point of M provided that the set of points $M - P$ is not connected. Cf. R. L. Moore, "Concerning the Cut Points of Continuous Curves and of other Closed and Connected Point Sets," *Proceedings of the National Academy of Sciences*, Vol. 9 (1923), pp. 101-106.

2. *Simple Cyclic Chains.*

Definition. A point set X will be called a *simple cyclic chain* of M between two cyclic elements E_1 and E_2 of M provided that X is connected and contains E_1 and E_2 and is the sum of the elements of some collection of the cyclic elements of M, and furthermore no proper connected subset of X contains E_1 and E_2 and is the sum of the elements of such a collection.

In my paper "Some Properties of Continuing curves," * I showed that if A and B are any two points of a continuous curve M, K denotes the set of all those points of M which separate † A and B in M, and t is any arc in M from A to B, then (1) $K+A+B$ is a closed set of points, and (2) each maximal segment S of $t-(K+A+B)$ determines a unique maximal cyclic curve of M which contains S.

THEOREM 1. *In order that the point set X should be a simple cyclic chain of a continuous curve M between two cyclic elements E_1 and E_2 of M it is necessary and sufficient that if t is any arc in M joining a point A of E_1 and a point B of E_2, and K denotes the set of all those points of M which separate A and B in M, then X is the set of points E_1+E_2+K+ all those maximal cyclic curves of M determined by the maximal segments of $t-(K+A+B)$.*

Proof. The condition is sufficient. Let X denote the set of points E_1+E_2+K+ all those maximal cyclic curves of M determined by the maximal segments of $t-(K+A+B)$ as described in the statement of this theorem. Then X is a simple cyclic chain of M from E_1 to E_2. Evidently X contains E_1 and E_2 and is the sum of the elements of a collection of cyclic elements of M. That X is connected follows immediately from the fact that X contains t and every cyclic element of M belonging to X has at least one point in common with t. No proper subset of X has these properties. For suppose, on the contrary, that some proper subset Y of X does have these properties. Then there exists at least one cyclic element C of M which is a subset of X but not of Y. Since each point of K separates A and B in M, and since Y is connected and contains both A and B, then Y must contain every point of K. Therefore C must be a maximal cyclic curve of M determined by some maximal segment S of $t-(K+A+B)$.

---

* *Bulletin of the American Mathematical Society*, Vol. 33 (1927), pp. 305-308.

† The point X of a connected set M is said to separate in M the two points A and B of M provided that $M-X$ is the sum of two mutually separated point sets M_a and M_b containing A and B respectively.

Let E and F be the endpoints of S. Since E and F belong to $K+A+B$, therefore they belong also to Y. Now Y can contain no internal point of C, for if it did it would necessarily contain all of C. Hence $Y \cdot C$ is countable. Since $Y \cdot C$ is countable and contains at least two points, it is not connected and hence can be expressed as the sum of two mutually separated point sets H_1 and H_2. For each point P of Y which does not belong to C, let T_p denote the maximal connected subset of $M - C$ which contains P. By theorem 2 of C. C., for each P, the set T_p has exactly one limit point Q which belongs to C. It is clear that for each P, the point Q must belong to Y and hence must belong either to H_1 or to H_2. Let G_1 be the set of all the points P of $Y - Y \cdot C$ such that the corresponding point Q belongs to the set H_1, and let G_2 be the set of all those points of $Y - Y \cdot C$ such that the corresponding point Q belongs to H_2. Then clearly $Y = H_1 + H_2 + G_1 + G_2$. Now no point of H_1 is a limit point of G_2. For suppose, on the contrary, that some point Z of H_1 is a limit point of G_2. Then since for no point P of G_2 is Z a limit point of the set T_p, it follows that there exists in G_2 an infinite sequence of points $P_1, P_2, P_3, \cdots$, having Z as its sequential limit point and such that the sets $T_{p_1}, T_{p_2}, T_{p_3}, \cdots$, are all distinct and have no points in common. But for each i, T_{p_i} has a limit point Q_i which belongs to H. Now for each point Q of the set $Q_1 + Q_2 + Q_3 + \cdots$, there are not more than a finite number of sets T_{p_i} which have Q for a limit point. For if the contrary were true, then since Z is the sequential limit point of the sequence $P_1, P_2, P_3, \cdots$, and since for any given number $\epsilon > 0$, not more than a finite number of the sets T_p are of diameter $> \epsilon$, it would follow that Q, a point of H_2, is a limit point of the set $P_1 + P_2 + P_3 + \cdots$, and therefore is identical with Z, a point of H_1, which is impossible. It follows, then, that the set of points $Q_1 + Q_2 + Q_3 + \cdots$ contains infinitely many distinct points, and since Z is a limit point of the sequence $P_1, P_2, P_3, \cdots$, it readily follows that Z is also a limit point of the set $Q_1 + Q_2 + Q_3 + \cdots$. But this is impossible, for Z belongs to H_1 and $Q_1 + Q_2 + Q_3 + \cdots$ belongs to H_2, and H_1 and H_2 are mutually separated sets. Thus the supposition that H_1 contains a limit point of G_2 leads to a contradiction. Now clearly no point of G_2 can be a limit point of H_1, for H_1 belongs to C, and G_2 belongs to $M - C$. Therefore H_1 and G_2 are mutually separated point sets. A similar argument shows that H_2 and G_1 are mutually separated sets. Then since H_1 and H_2 and also G_1 and G_2 are mutually separated, it follows that the sets $(G_1 + H_1)$ and $(G_2 + H_2)$ are mutually separated. But the sum of these two sets is Y, and Y, by

hypothesis, is connected. Thus the supposition that the set Y exists having the properties of the set X leads to a contradiction, and it follows that X is a simple cyclic chain of M from E_1 to E_2.

The condition is also necessary. For let Y be any simple cyclic chain of M between two cyclic elements E_1 and E_2 of M. Let X be the set of points $E_1 + E_2 + K +$ all those maximal cyclic curves of M determined by the maximal segments of $t - (K + A + B)$, where t is any arc in M from a point A of E_1 to a point B of E_2, and K is the set of all those points of M which separate A and B in M. It was shown above that X is a simple cyclic chain of M from E_1 to E_2. I shall show that Y must be identical with X. Suppose this is not so. Then X contains a cyclic element C of M which is not a subset of Y. Then by an argument identical with that given in the first part of the proof of this theorem, beginning with the sentence "Since each point of K separates A and B in M, . . . ," it is shown that this supposition leads to a contradiction. Hence Y is identical with X, and the theorem is proved.

The method of proof used under Theorem 1 yields immediately the following theorem.

THEOREM 2. *Let X be a simple cyclic chain between two cyclic elements E_1 and E_2 of a continuous curve M. Let C be any cyclic element of class (a) belonging to X and identical with neither E_1 nor E_2. Let E and F be the endpoints of the segment of t which determines C (see Theorem 1). Then $C - (E + F)$ separates E_1 and E_2 in M, i. e., $M - [C - (E + F)]$ is the sum of two mutually separated sets containing E_1 and E_2 respectively. Likewise $I(C)$ separates E_1 and E_2 in M.*

THEOREM 3. *If E_1 and E_2 are any two cyclic elements of a continuous curve M, then there exists in M one and only one simple cyclic chain from E_1 to E_2.*

Proof. Let t be any arc in M from a point A of E_1 to a point B of E_2. Let X be defined as in Theorem 1. Then by Theorem 1, X is a simple cyclic chain of M from E_1 to E_2. That X is the only simple cyclic chain of M from E_1 to E_2 follows at once from Theorem 1.

THEOREM 4. *A simple cyclic chain is a continuous curve.*

Proof. Let X be a simple cyclic chain of M between two cyclic elements E_1 and E_2 of a continuous curve M. Let A and B denote points of E_1 and E_2 respectively and t any arc in M from A to B. Then by Theorem 1, X is the

set of points $E_1 + E_2 + K +$ all those maximal cyclic curves of M determined by the maximal segments of $t - (K + A + B)$, where K is the set of all those points of M which separate A and B in M. By definition X is connected. And since, for any positive number ϵ, at most a finite number of the cyclic elements of M are of diameter $> \epsilon$, since t belongs to X, and each cyclic element of M which belongs to X contains a point of t, it readily follows that X is closed. Hence X is a continuum. The continuum X is also regular (connected im kleinen). For let P be a point of X and C any circle with P as center. Now if P is on t, there exists a circle C_1 with P as center and such that all those points of t lying within C_1 belong to some subarc of t lying within C and containing P. There exists a circle C_2 concentric with C_1 and such that every cyclic element of M which has a point within C_2 either lies wholly within C_1 or else it contains P. If N is any cyclic element lying wholly within C_1 and belonging to X, then since N has a point in common with t, it follows that N lies together with P in a connected subset of M lying within C. And since only a finite number of cyclic elements contain P and also some point without C, and each cyclic element is a continuous curve, it follows that there exists a circle C_3 concentric with and within C_2 and such that every point of X interior to C_3 lies together with P in a connected subset of M lying within C. Hence X is regular at P if P is on t. Now if P is nto on t, then P belongs to some maximal cyclic curve N of M, and is not a limit point of $X - N$. Hence in this case also X is regular at P. Thus we have shown that X is a continuous curve.

DEFINITION. If X is a simple cyclic chain between two cyclic elements E_1 and E_2 of a continuous curve M, the elements E_1 and E_2 will be called the *endelements of M*.

It may happen, in case E_1 or E_2 or both is a cut *point* of M, that some cyclic element of M different from E_1 belongs to X and contains E_1, and similarly for E_2. This will be the case when E_1 is not a limit point of the set K of points which separate in M a point A of E, and a point B of E_2. In this case, $E_1(E_2)$ is an endpoint of a maximal segment of $t - (K + A + B)$, where t is any arc in M from A to B, and clearly not more than one cyclic element of M different from $E_1(E_2)$ can belong to X and contain $E_1(E_2)$. It will be seen later, when the term endelement is defined for continuous curves in general, that any element of X which contains either E_1 or E_2 is also an endelement of X under the generalized definition. However, in this paper wherever we speak of an "endelement of a simple cyclic chain," we mean an endelement in the sense as above defined. All the cyclic elements of M which

belong to X and contain neither E_1 nor E_2 will be called the *interior* elements of the chain X.

THEOREM 5. *If X is a simple cyclic chain between two cyclic elements E_1 and E_2 of a continuous curve M, and C is any interior element of X, then $X - C = X_1 + X_2$, where X_1 and X_2 are mutually separated point sets, and (1) X_1 and X_2 are connected and contain points of E_1 and E_2 respectively, and (2) $X_1 + C$ and $X_2 + C$ are simple cyclic chains from E_1 to C and from E_2 to C respectively.*

Proof. Since C contains neither E_1 nor E_2, therefore, by Theorem 1, C is either a point of K or a maximal cyclic curve of M which contains at least one point of K, where K denotes the set of all those points of M which separate in M a point A of E_1 (an internal point of E_1 in case E_1 is an element of class (a)) and a point B of E_2. In either case it is easy to see that $X - C$ is not connected and that it can be expressed as the sum of two mutually seperated sets X_1 and X_2 containing points A and B of E_1 and E_2 respectively.

(1.) The sets X_1 and X_2 are connected. For suppose one of them, say X_1, is not connected. Then let Y denote the maximal connected subset of X_1 containing A. Now since, by Theorem 4, X is a continuous curve, it follows that $\overline{X}_2$ and $\overline{Y}$ are each the sum of the elements of some collection of cyclic elements of M, and that C contains a limit point of Y. Then since $X_2 + C$ is connected, $Y + X_2 + C$ is connected and contains E_1 and E_2 and is the sum of the elements of some collection of cyclic elements of M. But X is a simple cyclic chain of M from E_1 *to* E_2, and $Y + X_2 + C$ is a proper subset of X. This is contrary to the definition of a simple cyclic chain. It follows that X_1 is connected, and a similar proof shows that X_2 is connected.

(2.) The set of points $X_1 + C$ is a simple cyclic chain from E_1 to C. For $X_1 + C$ is connected and contains both E_1 and C and is the sum of the elements of some collection of cyclic elements of M. Furthermore $X_1 + C$ is irreducible with respect to these properties, for any assumption to the contrary leads immediately to the conclusion that X is reducible with respect to similar properties between E_1 and E_2, contrary to the fact that X is a simple cyclic chain from E_1 to E_2. Therefore $X_1 + C$ is a simple cyclic chain from E_1 to C, and a similar proof shows that $X_2 + C$ is a simple cyclic chain from E_2 to C. This completes the proof of the theorem.

THEOREM 6. *A simple cyclic chain X of a continuous curve M is not disconnected by the omission of either one or both of its endelements, E_1*

and E_2 nor by the omission of either one or both of the cyclic elements of M (if such exist) which belong to X and contain E_1 or E_2.

Proof. Let X_1 denote the set of points $X - E_1$. Then by almost the same argument as given in part (1) of the proof of Theorem 5, it is shown that X_1 (if it exists, of course) is connected. Similarly it is shown that $X - E_2$ is connected. Now let H denote the set of points $X - (E_1 + E_2)$. Suppose, contrary to this theorem, that H is not connected. Then since, by Theorem 4, X is a continuous curve, it follows that some maximal connected subset Y of H has at least one limit point in each of the sets E_1 and E_2. Then $E_1 + E_2 + Y$ is a connected point set which contains E_1 and E_2 and is the sum of the elements of some collection of the cyclic elements of M. But $E_1 + E_2 + Y$ is a proper subset of X, and by hypothesis X is a simple cyclic chain from E_1 to E_2. Thus the supposition that H is not connected leads to a contradiction.

Now if in the argument just given we replace E_1 and E_2 by E_1^0 and E_2^0, where E_1^0 and E_2^0 respectively denote the cyclic elements of M (if any such exist) which lie in X and contain E_1 and E_2 respectively, we arrive at the conclusion that $X - E_1^0$, $X - E_2^0$, and $X - (E_1^0 + E_2^0)$ are connected point sets. This completes the proof of the theorem.

The close relationship between a simple cyclic chain between two cyclic elements E_1 and E_2 of a continuous curve M and a simple continuous arc is at once apparent from the foregoing propositions and discussion. The definition given above for a simple cyclic chain parallels Lennes'* definition of a simple continuous arc. However, the analogy between the two sets is not complete, due to the fact that two cyclic elements of a continuous curve may have a point in common without being identical. In the case of an arc, it is true† that a necessary and sufficient condition that a bounded continuum be a simple continuous arc between two of its points A and B is that it be disconnected by the omission of any one of its points different from A and from B. Theorem 5 shows that a simple cyclic chain is disconnected by the omission of any one of its interior elements. It is not sufficient, however, in order that a bounded continuum which is the sum of the elements of a collection of cyclic elements of a continuous curve M should

---

* N. J. Lennes, "Curves in Non-Metrical Analysis Situs with an Application in the Calculus of Variations," *American Journal of Mathematics*, Vol. 33 (1911), p. 308.

† Cf. R. L. Moore, "Concerning Simple Continuous Curves," *Transactions of the American Mathematical Society*, Vo. 21 (1920), pp. 333-347.

be a simple cyclic chain between two of its elements E_1 and E_2 that it should be disconnected by the omission of any one of its cyclic elements different from E_1 and E_2. For let E_1 and E_2 be two circles each without the other except for P and which are tangent to each other at the point P. Let C_1 and C_2 be circles lying within E_1 and E_2 respectively except for the point P and which are also tangent at P. Let M denote the continuous curve $E_1 + E_2 + C_1 + C_2$. The cyclic elements of M are E_1, E_2, C_1, C_2, and P, and the omission of *any* cyclic element of M disconnects M. But M is not a simple cyclic chain from E_1 to E_2.

3. *Some Acyclic Properties of Continuous Curves in General.* In this section it is purposed to show that many of the characteristic properties of acyclic continuous curves are properties of any continuous curve which is regarded as being composed of its cyclic elements. It is well known that a simple closed curve may be defined as a point set which is the sum of two simple continuous arcs having in common only their two endpoints. In strict analogy, let us define a *closed curve of cyclic elements* of a continuous curve M as a point set which is the sum of two *different* simple cyclic chains of M which have in common only their two endelements. Then Theorem 3 shows that there does not exist any closed curve of cyclic elements of a continuous curve. Thus we get the following interesting theorem.

THEOREM 7. *Every continuous curve is acyclic with respect to its cyclic elements.*

Thus it is apparent that so long as we are dealing with a single continuous curve, a study of the *closed curves* of its cyclic elements is vacuous. However, a consideration of such curves in the study of two or more continuous curves having points in common will probably lead to interesting conclusions. It is not the purpose of this paper to proceed any further in this direction than merely to mention the possibility of further research.

DEFINITION. A subset H of a continuous curve M will be said to be *cyclic chainwise connected* provided every two cyclic elements of M which belong to H can be joined by a simple cyclic chain of M which is a subset of H.

THEOREM 8. *If the sum H of the elements of a collection of cyclic elements of a continuous curve M is connected, then H is both arcwise connected and cyclic chainwise connected. Furthermore H contains every simple continuous arc of M which joins two points of H.*

Proof. Let H be any connected point set which is the sum of the elements of some collection of the cyclic elements of M, and let A and B be any two points of H. Let E_1 and E_2 be cyclic elements of M containing A and B respectively and belonging to H, and let K be the set of all those points of M which separate A and B in M. By Theorem 3, there exists in M a simple cyclic chain X from E_1 to E_2. Then X must be a subset of H. For suppose it is not. Then it contains a cyclic element C which is not a subset of H. Since H is connected and contains A and B, then it must contain every point of K; and therefore, by Theorem 1, C is a maximal cyclic curve of M which is determined by some maximal segment of $t-(K+A+B)$, where t is any arc in M from A to B. Now no internal point of C can belong to H. For if it did, then it would belong to some cyclic element of M belonging to H, and this element must be identical with C, for no two cyclic elements can have an internal point of either in common without being identical. Hence $H \cdot I(C) = 0$. But this is contrary to Theorem 2, which states that $I(C)$ separates E_1 and E_2 in M, whereas H is a connected subset of $M - I(C)$ containing both E_1 and E_2. Thus the supposition that X is not a subset of H leads to a contradiction. Then since H contains X and, by Theorem 1, X contains t, therefore H contains an arc t from A to B. Hence H is arcwise connected.

That H is cyclic chainwise connected follows by the same argument as just given, where we let E_1 and E_2 be any two cyclic elements of M belonging to H. The same argument shows that H contains every arc in M which joins any two points of H, for A and B were any two points of H and t any arc in M from A to B, and it was shown that H contains t.

It is interesting to note that the following theorem of C. M. Cleveland's *: *Every connected subset of the set of all the cut points of a continuous curve is arcwise connected,* is a corollary to Theorem 8. This is evidently true because each cut point of a continuous curve is, by definition, a cyclic element of that curve.

In my paper *Concerning certain types of continuous curves* † I showed that a necessary and sufficient condition that a bounded continuum M should be an acyclic continuous curve is that every connected subset of M should be uniformly connected im kleinen. I shall now prove the following related theorem.

---

* Cf. an abstract of his paper "Concerning Cut Points of a Continuous Curve" in the *Bulletin of the American Mathematical Society*, Vol. 32 (1926), p. 420.

† *Proceedings of the National Academy of Sciences*, Vol. 12 (1926), pp. 761-767, Theorem 2.

THEOREM 9. *If the sum H of the elements of a collection of the cyclic elements of a continuous curve M is connected, then H is uniformly connected im kleinen.*

Proof. Suppose, contrary to this theorem, that H is not uniformly connected im kleinen. Then there exists a positive number ϵ such that H contains two infinite sequences of points $X_1, X_2, X_3, \cdots$, and $Y_1, Y_2, Y_3, \cdots$, having the property that for each positive integer n, the distance from X_n to Y_n is less than 1/n, but such that for no integer n can X_n be joined to Y_n by a connected subset of H of diameter less than ϵ. But since M is uniformly connected im kleinen, there exists an integer k such that X_k and Y_k can be joined in M by an arc X_kY_k of diameter less than ϵ. But by Theorem 8, the arc X_kY_k, since it joins two points of H, must belong to H. Thus the supposition that H is not uniformly connected im kleinen leads to a contradiction, and the theorem is proved.

THEOREM 10. *If the sum H of the elements of a collection of cyclic elements of a continuous curve is a continuum, then H is a continuous curve.*

Theorem 10 is a corollary to Theorem 9.

THEOREM 11. *If the sum H of a collection of the cyclic elements of a continuous curve M is connected, then each point of $\overline{H} - H$ is a cyclic element of M and $\overline{H}$ is a continuous curve.*

Proof. Let P be any point of $\overline{H}-H$. Then P is either a cut point or an endpoint of M. For suppose it is neither. Then by Theorem 7 of C. C. it follows that P is an internal point of some maximal cyclic curve C of M. Now H contains no point of $I(C)$, for if it did it would contain all of $I(C)$ and therefore would contain P. Let N denote the maximal connected subset of $M - I(C)$ which contains H. Then N is a continuous curve having exactly one point Q in common with C. For since H is connected and $C - I(C)$ is countable, therefore H contains a point X in common with $M - C$. Let N_x denote the maximal connected subset of $M - C$ containing X. Then by Theorem 2 of C. C., C contains exactly one limit point Q of N_x, and clearly Q is a cut point of M. Let G denote the collection of all the maximal connected subsets of $M - C$ which have Q as a limit point, and let N_0 denote the sum of all the point sets of the collection G. Since by a theorem of W. L. Ayres',* at most a finite number of the elements of G are of diameter

* "Concerning Continuous Curves and Correspondences," *Annals of Mathematics*, Vol. 28 (1927), Theorem 1. The special case of this theorem here used holds in n dimensions.

greater than any preassigned positive number, it follows at once that $N_0 + Q$ is closed and, in fact, is a continuous curve. Clearly $N_0 + Q$ is a subset of N. It must be identical with N. For suppose N contains a point Y not belonging to $N_0 + Q$. It is easy to see that every point of N belongs to a cyclic element of M which is contained in N. Hence, by Theorem 8, N contains an arc XY. On the arc XY, in the order from Y to X, it is readily seen that Q must be the first point belonging to $N_0 + Q$. By the definition of the set N_0, it follows that the segment YQ of the arc XY must contain at least one point of C; but since it can contain only a countable number of points of C, it follows that it contains some segment S whose endpoints are on C but which contains no point of C. This is contrary to Theorem 2 of C. C., which states that no connected subset of $M - C$ has more than one limit point in C. Thus the supposition that N is not identical with $N_0 + Q$ leads to a contradiction. Hence N is a continuous curve having in common with C only the point Q. But H is a subset of N, and by supposition P, an internal point of C, is a limit point of H. Clearly this is impossible. Thus the supposition that P is neither an endpoint nor a cut point of M leads to a contradiction.

Now since every point of $\overline{H} - H$ is either an endpoint or a cut point of M, and since every such point is, by definition, a cyclic element of M, it follows that there exists a collection of cyclic elements of M whose sum is H. Therefore, since $\overline{H}$ is a continuum, by Theorem 10, it is a continuous curve. This completes the proof of our theorem.

4. *Nodes and Endelements.* A cyclic element E of a continuous curve M will be called a *node* of M provided E neither is itself a cut point of M nor contains more than one cut point of M. Following a definition of an *endpoint* of a continuous curve given in my thesis,* let us define an *endelement* of a continuous curve as follows. The cyclic element E of a continuous curve M will be called an *endelement* of M provided that if X is any simple cyclic chain in M having E as one of its endelements, then no connected subset of $M - X$ has a limit point in E. It is clear from these definitions that every *endpoint* of M is both a *node* and an *endelement* of M and, indeed, that every endelement of M is also a node of M. However, not all nodes of M are necessarily endelements of M. For let $M = C + I$, where I is the interval $(-1, 1)$ and C is the circle $x^2 + (y-1)^2 = 1$. Then C is a node of M; but

* G. T. Whyburn, "Concerning Continua in the Plane," *loc. cit.*, p. 386. An endpoint of a continuous curve M is there defined as a point P of M such that if t is any arc of M having P as one of its extremities, then P is not a limit point of any connected subset of $M - t$.

it is not an endelement of M, because C contains a limit point of the segment $(-1, 0)$, a connected subset of M minus the simple cyclic chain consisting of the interval $(1, 0)$ plus C.

THEOREM 12. *In order that a cyclic element of a continuous curve M should be a node of M it is necessary and sufficient that it should be either an endpoint of M or a maximal cyclic curve C of M such that $M - I(C)$ is connected (or vacuous).*

Proof. The condition is necessary. For let C be a node of M which is not an endpoint of M. Then since by definition C is not a cut point of M, therefore it is a maximal cyclic curve of M. Now unless C is identical with M, in which case our theorem is obvious, then C contains at least one cut point P of M. By the definition of a node it follows that P is the only cut point of M which belongs to C. Hence P is a limit point of every maximal connected subset of $M - C$, and therefore the sum H of all the maximal connected subsets of $M - C$ plus P is connected. But $H = M - C = M - P - (C - P)$, and $C - P = I(C)$. Therefore $H + P = M - I(C)$, and hence $M - I(C)$ is connected.

The condition is also sufficient. For let C be any cyclic element of M which is either an endpoint of M or a maximal cyclic curve of M such that $M - I(C)$ is connected. Now if C is an endpoint of M, obviously it must be a node. So let us suppose C is a maximal cyclic curve of M. Now it was shown in the proof of Theorem 11 that if C is any maximal cyclic curve of M, then every maximal connected subset of $M - I(C)$ is a continuous curve having one and only one point in common with C. And since, in this case, $M - I(C)$ is connected, therefore it is a continuous curve having exactly one point P in common with C. Hence $I(C) = C - P$, and P is the only cut point of M which belongs to C. Therefore C satisfies all the requirements of the definition of a node of M, and the theorem is proved.

THEOREM 13. *In order that a cyclic element E of a continuous curve M should be a node of M it is necessary and sufficient that no simple cyclic chain in M should have E as one of its interior elements.*

Proof. The condition is necessary. For suppose, on the contrary, that a node E of M is an interior element of some simple cyclic chain X in M between two cyclic elements E_1 and E_2 of M. Now by the definition of an interior element of X, E contains neither E_1 nor E_2. Hence there exist points A and B in E_1 and E_2 respectively which do not belong to E. Let t be an arc

in M from A to B, and let K denote the set of all those points of M which separate A and B in M. By Theorem 1, X is the set of points $E_1 + E_2 + K +$ all those maximal cyclic curves of M determined by the maximal segments of $t - (K + A + B)$. Since E is a node, it cannot belong to K, and it does not belong to either E_1 or E_2. Hence E is a maximal cyclic curve of M determined by some maximal segment S of $t - (K + A + B)$. Let F and G denote the endpoints of S. Since F and G belong to E, therefore A, B, F, and G are all distinct points. But then F and G are points of K and hence cut points of M. This is contrary to the fact that a node can contain at most one cut point of M. Thus the supposition that the condition of this theorem is not necessary leads to a contradiction.

The condition is also sufficient. For let E be a cyclic element of M which is not an interior element of any simple cyclic chain in M. Then E is not a cut point of M. For suppose it is. Then E separates in M some two points A and B of M. Then by Theorem 1, E is an element of the simple cyclic chain X which joins a cyclic element E_1 containing A and an element E_2 containing B. And as E contains neither E_1 nor E_2, therefore it is an interior element of X, contrary to hypothesis. Now since E is not a cut point of M, it must be either an endpoint of M or a maximal cyclic curve of M. In the former case our theorem is obvious. So let us suppose that E is a maximal cyclic curve of M. Then $M - I(E)$ is connected. For suppose it is not. Then there exist two distinct maximal connected subsets N_1 and N_2 of $M - I(E)$. It was shown in the proof of Theorem 11 that N_1 and N_2 are continuous curves containing points P_1 and P_2 respectively in common with E. Let A and B be points of $N_1 - P_1$ and $N_2 - P_2$ respectively. Let E_1 and E_2 be cyclic elements of M which contain A and B respectively. By Theorem 3, M contains a simple cyclic chain X from E_1 to E_2. Let t be an arc in M from A to B and K the set of all those points of M which separate A and B in M. Then P_1 and P_2 belong to K, and the interval $P_1 P_2$ of t belongs to E. Hence by Theorem 1 it follows that E is an element of X, and since E contains neither A nor B, it is therefore an interior element of X. Thus the supposition that $M - I(E)$ is not connected leads to a contradiction. Hence $M - I(E)$ is connected and, by Theorem 12, E is a node of M.

THEOREM 14. *If the sum H of the elements of a collection G of cyclic elements of a continuous curve M is a continuum, and H itself has more than one cyclic element, then H contains at least two nodes of itself.*

Proof. Every cyclic element of H is a cyclic element also of M. For every maximal cyclic curve of H is a maximal cyclic curve also of M, every

cut point of H is a cut point also of M, and every endpoint of H is a cut point or an endpoint of M.

Let us suppose, contrary to this theorem, that H does not contain as many as two nodes of itself. Then since H, by hypothesis, has more than one cyclic element, it follows by Theorem 12 and the definition of a node that H contains a cyclic element C which is either a cut point of H or a maximal cyclic curve of H such that $H - I(C)$ is not connected. Let E denote C or $I(C)$ according as C is a cut point or a maximal cyclic curve of H. Then there exist at least two distinct maximal connected subsets H_1 and H_2 of $H - E$. Since, by supposition, H has not more than one node, then either $H_1 + C$ or $H_2 + C$, say $H_1 + C$, contains no node of H. Now either H_1 has no point whatever in common with C or else (see proof of Theorem 11), H_1 is a continuous curve containing exactly one point P in common with C. In the former case, C is a cut point of H and a limit point of H_1. In this case let $P = C$. In either case H_1 contains a countable set of points $P_1, P_2, P_3, \cdots$,— all distinct from P, and such that every point of H_1 is a limit point of this set. For each positive integer i, let $C(P_i)$ denote a cyclic element of H which either is identical with P_i or contains P_i, and let E_i denote $C(P_i)$ or $I[C(P_i)]$ according as $C(P_i)$ is a cut point or a maximal cyclic curve of H. Since H_1 contains no node of H, then by Theorem 12 and the definition of cyclic element it follows that for every positive integer i, $H - E_i$ is not connected. It is readily seen that for no integer $i > 0$ does E_i contain the point P. For every integer $i > 0$, let H_{ip} denote the maximal connected subset of $H - E_i$ which contains P. There exists an integer n_1 which is the least integer such that P_{n_1} belongs to $H - [H_{1p} + C(P_1)]$. Now E_{n_1} contains no point whatever in common with $H_{1p} + C(P_1)$. Therefore, since $H_{1p} + C(P_1)$ is connected, it is a subset of H_{n_1p}. There exists an integer n_2 which is the least integer such that P_{n_2} belongs to $H - [H_{n_1p} + C(P_{n_1})]$ Just as above it follows that $H_{n_1p} + C(P_{n_1})$ is a subset of H_{n_2p}. There exists a least integer n_3 such that P_{n_3} belongs to $H - [H_{n_2p} + C(P_{n_2})]$, and so on. This process may be continued indefinitely, giving an infinite sequence of distinct points P_{n_1}, P_{n_2}, P_{n_3}, $\cdots$, and an infinite sequence of connected sets H_{n_1p}, H_{n_2p}, $\cdots$, such that for every positive integer i, $H_{n_ip} + C(P_{n_i})$ is a subset of $H_{n_{i+1}p}$.

The set of points $P_{n_1} + P_{n_2} + P_{n_3} + \cdots$ has at least one limit point Q which belongs to H_1. It is easily seen from the method of selection of the points P_{n_i} that Q belongs to none of the sets H_{n_ip}. The point Q is not a cut point of H. For suppose it is. Then $H - Q = N_1 + N_2$ where N_1 and N_2 are mutually separated sets, one of which, say N_1, contains $\sum H_{n_ip}$. But N_2 must contain some point, say P_k, of the sequence $P_1, P_2, P_3, \cdots$. Clearly

this is impossible, because the above process of selection of the points P_{n_1}, P_{n_2}, $\cdots$ would, after at most $k+1$ steps, throw the point P_k into one of the sets $H_{n_i p}$, contrary to what has just been shown. Hence Q is not a cut point of H. Therefore, by a theorem of the author's,* Q is either an endpoint of H or else it is an internal point of some maximal cyclic curve F of H. The point Q is not an endpoint of H, because by supposition H_1 contains no node of H. But neither is P an internal point of any maximal cyclic curve F of H, because Q is a limit point of $\sum H_{n_i p}$ and $\sum H_{n_i p}$ is connected and a subset of $H - I(F)$, and therefore no internal point of F is a limit point of $\sum H_{n_i p}$. Thus the supposition that H does not contain two nodes of itself leads to a contradiction, and the theorem is proved.

Mazurkiewicz has shown † that if M is an acyclic continuous curve and t is any arc of M, then M contains a maximal arc which contains t, *i. e.*, M contains an arc T containing t and which is not a proper subset of any other arc of M. This theorem is evidently not true for any continuous curve which contains a simple closed curve. I shall now prove the following theorem which gives a similar result for *any continuous curve*, provided we use simple cyclic chains instead of simple continuous arcs.

THEOREM 15. *If X is any simple cyclic chain of a continuous curve M, then M contains a maximal simple cyclic chain which contains X,* i. e., *a simple cyclic chain which contains X and is not a proper subset of any other simple cyclic chain of M.*

Proof. Let E_1 and E_2 denote the endelements of X. Now if E_1 is not a node of M, then, by Theorem 12, it is either a cut point of M or a maximal cyclic curve of M such that $M - I(E_1)$ is not connected. In either case, it follows with the aid of Theorem 6 that there exists a maximal connected subset H_1 of $M - X$ which has exactly one limit point P_1 in E_1, where P_1 is not a limit point of $X - E_1$ if E_1 is an element of class (a). Likewise if E_2 is not a node, then there exists a maximal connected subset H_2 of $M - X$ which has exactly one limit point P_2 in E_2, where P_2 is not a limit point of $X - E_2$ if E_2 is an element of class (a). Now if either of the continuous curves $H_1 + P_1$ and $H_2 + P_2$ has only one cyclic element, it is clear that that element must be a node of M. And if either of these curves contains more than one cyclic element, then by Theorem 14 it follows that that curve contains at least two

* "Concerning Continua in the Plane," *loc. cit.*, Theorem 22.

† S. Mazurkiewicz, "Un théorème sur les lignes de Jordan," *Fundamenta Mathematicae*, Vol. 2 (1921), p. 129.

nodes of itself; and since at least one of these nodes does not contain the point $P_i (i = 1, 2)$, it is readily seen that in each case at least one of these nodes must be a node also of M. Hence in any case if E_1 and E_2 themselves are not nodes of M, then $H_1 + P_1$ and $H_2 + P_2$ contain nodes N_1 and N_2 respectively of M. If either E_1, or E_2, or both, is a node of M, then let $N_1 = E_1$, $N_2 = E_2$, or both. By Theorem 3, M contains one simple cyclic chain T from N_1 to N_2. With the aid of Theorem 1 it is easy to see that T must contain X, and since N_1 and N_2 are nodes of M, therefore, by Theorem 13, they are the endelements of every simple cyclic chain of M which contains them. Hence by Theorem 3 it follows that T is identical with every simple cyclic chain of M which contains T. Therefore T is a maximal simple cyclic chain in M containing X.

R. L. Wilder has shown * that a necessary and sufficient condition that a point of an acyclic continuous curve M should be an endpoint of M is that it should be a non-cut point of M. This theorem obviously does not hold for any continuous curve which contains a simple closed curve. However, I shall now show that it does hold for any continuous curve provided we use cyclic elements instead of points.

THEOREM 16. *In order that the cyclic element E of a continuous curve M should be an endelement of M it is necessary and sufficient that it should be a non-cut element of M.*

Proof. The condition is necessary. For suppose, on the contrary, that an endelement E of M is a cut element of M. Then since E is a node of M, it readily follows that there exist two maximal connected subsets H and N of $M - E$ and a point P of E which is a limit point of each of the sets H and N. Let x be a point of H and F a cyclic element of M containing x. By Theorem 3, there exists in M a simple cyclic chain X from F to E. Since, by Theorem 6, $X - E$ is connected, therefore it is a subset of H. Hence N contains no point in common with X. But the point P of E is a limit point of N, which is impossible, since E is an endelement of M. Thus the supposition that the condition of this theorem is not necessary leads to a contradiction.

The condition is also sufficient. For let E be an element of M such that $M - E$ is connected, and suppose, contrary to this theorem, that E is not an endelement of M. Then there exists in M a simple cyclic chain X from E to some other element F of M and a maximal connected subset N of $M - X$ having a limit point P in E. Now since $M - E$ is connected, by Theorem 2

* "Concerning Continuous Curves," *loc. cit.*, Theorem 7.

of C. C., P is the only limit point of $M - E$ which belongs to E. There exists in X an arc AP, where A is some point of $X - E$. By a theorem of R. L. Wilder's,* there exists an arc BP such that $BP - P$ belongs to N. By a theorem of R. L. Moore's, † $M - E$ contains an arc AB. The sum of the arcs AB, AP, and BP contains a simple closed curve J which contains a segment of each of the arcs AP and BP. But then by Theorem 1, J must belong to X, and BP belongs to N and therefore contains no point of X. Thus the supposition that E is not an endelement of M leads to a contradiction.

Theorem 16 gives immediately the following theorem:

THEOREM 17. *In order that a cyclic element E of a continuous curve M containing more than one cyclic element should be an endelement of M it is necessary and sufficient that it should be either an endpoint of M or a maximal cyclic curve C of M which contains exactly one cut point P of M and such that $M - P = (C - P) + S$, where $C - P$ and S are mutually separated and connected point sets.*

It has been shown by the author ‡ and also by Menger ‡ that the sum of all the endpoints of a continuous curve is a totally disconnected point set. It is not necessarily true, however, that the sum of all the endelements of a continuous curve contains no connected set containing more than one endelement of M. For let $M = C_1 + C_2$, where C_1 and C_2 are circles tangent to each other at a point P. Then both C_1 and C_2 are endelements of M and their sum is connected. The following two theorems are of interest in this connection.

THEOREM 18. *In order that two different endelements E_1 and E_2 of a continuous curve M should have a common point it is necessary and sufficient that $E_1 + E_2 = M$.*

Proof. The condition is sufficient. For E_1 and E_2 are closed point sets, and if their sum is M, a connected point set, evidently they must have a common point.

The condition is also necessary. For suppose the endelements E_1 and E_2 of a continuous curve M have a common point P, and suppose, contrary to

---

* "Concerning Continuous Curves," *loc. cit.*, Theorem 1.

† "Concerning Continuous Curves in the Plane," *Mathematische Zeitschrift*, Vol. 15 (1922), pp. 254-260, Theorem 1.

‡ G. T. Whyburn, "Concerning Continua in the Plane," *loc. cit.*, Theorem 21; K. Menger, "Grundzüge einer Theorie der Kurven," *Mathematische Annalen*, Vol. 95 (1925), pp. 272-306.

this theorem, that $E_1 + E_2 \neq M$. Then there exists a maximal connected subset N of $M - (E_1 + E_2)$. But since E_1 and E_2 have a common point, it is clear that if X denotes the simple cyclic chain in M from E_1 to E_2, then $E_1 + E_2 = X$. Hence $M - (E_1 + E_2) = M - X$. But X contains at least one limit point P of N. The point P must belong either to E_1 or to E_2. But this is impossible since both E_1 and E_2 are endelements of M, and hence, by definition, neither contains a limit point of any connected subset of $M - X$.

THEOREM 19. *If the continuous curve M has more than three cyclic elements, or has two mutually exclusive elements, then the sum of all the endelements of M contains no connected point set which contains more than one endelement of M.*

Proof. If M has more than three cyclic elements, then no two endelements of M can have a common point. For if two endelements E_1 and E_2 of M have a common point P, then by Theorem 18, $E_1 + E_2 = M$, and M has only the three cyclic elements E_1, E_2, and P. Furthermore, if M has two mutually exclusive cyclic elements, then again no two endelements of M can have a common point. For if two such elements E_1 and E_2 had a common point P, then by Theorem 18, $E_1 + E_2 = M$, and M has only the three cyclic elements E_1, E_2, and P, every two of which have the point P in common. Hence, in either case, no two endelements of M can have a common point.

Now suppose, contrary to this theorem, that the sum of all the endelements of M contains a connected set H which contains as many as two endelements E_1 and E_2 of M. Let N denote the sum of all the endelements of M which contain at least one point of H. Then by Theorem 8, N contains a simple cyclic chain X of M from E_1 to E_2. But since E_1 and E_2 have no common point, it follows that there exists an element E of M which is an interior element of X. But E belongs to N and hence is an endelement of M. This is contradictory to Theorem 13 and the fact that every endelement of M is also a node of M. Thus the supposition that Theorem 19 is not true leads to a contradiction.

5. *Some Theorems on Continuous Curves.*

THEOREM 20. *Let P be a cut point of a continuous curve M which is an irregular* point of some subcontinuum N_0 of M. Then P belongs to some simple closed curve in M.*

---

* The point P of a point set N is an irregular point of N provided N is not connected im kleinen at P.

Proof. Since for each positive number ϵ, not more than a finite number of maximal connected subsets of $M - P$ are of diameter greater than ϵ, it readily follows that there exists a maximal connected subset H of $M - P$ such that P is an irregular point of the continuum $N_0 \cdot (H + P)$. Let N denote the continuum $N_0 \cdot (H + P)$. Now P is not a cut point of the continuous curve $H + P$. Therefore, by a theorem of the author's,* P either is an endpoint of $H + P$ or belongs to some simple closed curve in $H + P$. Let us suppose that P is an endpoint of $H + P$. Then since, as proved by H. M. Gehman,† for the case of a continuous curve, our definitions for the term *endpoint* as applied to continuous curves and for the same term as applied to continua in general are equivalent, it follows that P is an endpoint of the *continuum* $H + P$ and hence is an endpoint of the sub-continuum N of $H + P$. Now P obviously is accessible from some complementary domain of N, namely, from the one which contains some maximal connected subset of $M - (H + P)$. Therefore, by a theorem of mine,‡ it follows that P is a regular point of N, contrary to hypothesis. Thus the supposition that P is an endpoint of $H + P$ leads to a contradiction, and it follows that P must belong to some simple curve in $H + P$. This completes the proof.

THEOREM 21. *If G denotes the set of all those cut points $[P]$ of a continuous curve M such that P is an irregular point of some subcontinuum of M, then G is countable.*

Theorem 21 is an immediate consequence of Theorem 20 and of the theorem of the author's § which states that the set of all the cut points of a continuous curve M which belong to the sum of all the simple closed curves contained in M is countable.

I have recently succeeded in establishing the conclusion of Theorem 21 for *any plane continuum* M. Since the proof of this much more general proposition is of considerable length and is foreign to the material presented in this paper, it will be reserved for a later paper.

THEOREM 22. *In order that the point P of a continuous curve M should belong to no simple closed curve in M it is necessary and sufficient,*

---

* G. T. Whyburn, "Concerning continua in the plane," *loc. cit.*, Theorem 22.

† "Concerning End Points of Continuous Curves and Other Continua," *Transactions of the American Mathematical Society*, Vol. 30 (1928), pp. 63-84.

‡ *Loc. cit.*, Theorem 30.

§ *Loc. cit.*, Theorem 29.

that it should be an endpoint of every continuous curve obtained by adding it to some maximal connected subset of $M - P$.

Proof. The condition is necessary. For suppose P belongs to no simple closed curve in M, and let N be any continuous curve obtained by adding P to some maximal connected subset H of $M - P$. Then P belongs to no simple closed curve in N. And since $H = N - P$ is connected, therefore P is not a cut point of N. But every continuous curve is the sum of its cut points, endpoints, and simple closed curves.* Therefore P is an endpoint of N.

The condition is also sufficient. For suppose P is an endpoint of every continuous curve obtained by adding it to some maximal connected subset of $M - P$, but suppose, contrary to this theorem, that P belongs to some simple closed curve J in M. Then since $J - P$ is connected, it must belong to some maximal connected subset H of $M - P$, and clearly P is not an endpoint of $H + P$, contrary to hypothesis. Thus the supposition that P belongs to some simple closed curve in M leads to a contradiction.

THEOREM 23. *Let P be any endpoint of a continuous curve M. Then there exists in M an infinite sequence of points $X_1, X_2, X_3, \cdots$, having P as its sequential limit point and such that if A is any point whatever of $M - P$, then there exists an integer k_a such that if i is any integer greater than k_a, then X_i separates A from P in M.*

Proof. Let A_0 be any point of $M - P$, t_0 any arc in M from A_0 to P, and K_0 the set of all those points in M which separate A_0 and P in M. Then P is a limit point of K_0. For if not, there exists † a maximal cyclic curve of M which contains P, contrary to the fact that P is an endpoint of M. Hence there exists a sequence of points $X_1, X_2, X_3, \cdots$, all belonging to K_0, and having P as its sequential limit point. Now let A be any other point of $M - P$, t any arc in M from A to P, and K the set of all those points of M which separate A from P in M. On t, in the order from A to P, let E be the first point belonging to t_0. Since P is an endpoint of M, E must be different from P. There exists an integer k_a such that if i is any integer greater than k_a, then X_i belongs to the segment EP of t_0. Let j be any integer greater than k_a. Then X_j must belong to K. For $M - X_j = M_{A_0} + M_P$, by hypothesis, where M_{A_0} and M_P are mutually separated sets containing A_0 and P respectively. Now the arc AEA_0 does not contain X_j, for X_j belongs to the segment

* G. T. Whyburn, *loc. cit.*, Theorem 22.

† G. T. Whyburn, "Some Properties of Continuous Curves," *loc. cit.*, Theorem 2.

EP of t_0. Therefore the arc AEA_0, and hence also the point A, belongs to M_{A_0}. Hence X_j separates A from P in M and therefore belongs to K. This completes the proof of the theorem.

Theorem 23 is a generalization of H. M. Gehman's * theorem that every endpoint of a continuous curve is a limit point of cut points of that curve.

THEOREM 24. *In order that the continuous curve M should fail to separate the plane it is necessary and sufficient that every maximal cyclic curve of M should be a simple closed curve plus its interior.*

Proof. The condition is necessary. For suppose M does not separate the plane. Let C be any maximal cyclic curve of M, and let J be the boundary of the unbounded complementary domain of C. By Theorem 10 of C. C., J is a simple closed curve. Since M does not separate the plane, clearly every point within J belongs to M and also to C. That J plus its interior must be identical with C follows immediately from the fact that J is the boundary of the unbounded complementary of C.

The condition is also sufficient. For suppose every maximal cyclic curve of a continuous M is a simple closed curve plus its interior, and suppose, contrary to this theorem, that M separates the plane. Then there exists at least one bounded complementary domain R of M. Let J denote the boundary of the unbounded complementary domain of the boundary of R. By a theorem of R. L. Moore's,† J is a simple closed curve whose interior I contains R. By Theorem 3 of C. C., M contains a maximal cyclic curve C which contains J. By hypothesis there exists a simple closed curve K such that if D is the interior of K, then $K + D = C$. But C contains J and hence must also contain I, the interior of J. But I contains R, a complementary domain of M. Thus the supposition that M separates the plane leads to a contradiction.

THEOREM 25. *In order that a maximal cyclic curve C of a continuous curve M should be a simple closed curve it is necessary and sufficient that C should be the common boundary of two complementary domains of M.*

Proof. That the condition is sufficient is a consequence of a theorem of R. L. Moore's.‡ I shall show that it is necessary. Let C be a simple closed curve in M which is a maximal cyclic curve of M. If the interior I of C is a

---

* H. M. Gehman, *loc. cit.* It is interesting to note that the condition in Theorem 23 characterizes an endpoint of a continuous curve, whereas that in Gehman's theorem does not.

† R. L. Moore, "Concerning Continuous Curves in the Plane," *Mathematische Zeitschrift*, Vol. 15 (1922), pp. 254-260, Theorem 4.

‡ *Loc. cit.*

complementary domain of M, then let $I = R$. And if I is not complementary to M, then there exists in I a maximal connected subset H_0 of $M - C$ which has exactly one limit point P in C. Let H be the sum of all the maximal connected subsets of $M - C$ which lie in I and have P for a limit point. Let B be a point of $C - P$ and A_0 a point of H. There exists an arc A_0B such that $A_0B - B$ belongs to I. In the order from B to A_0 on the arc A_0B, let A be the first point belonging to the continuum $H + P$. Then A belongs to H, and there exists an interval AX of the arc AB such that $AX - A$ contains no point whatever of M. Now let R denote the complementary domain of M which contains the point X. Then every point of C belongs to the boundary of R. For suppose some point Y of C does not belong to the boundary of R. Then by a theorem of R. L. Moore's,* there exists in M a simple closed curve J which either encloses R and has Y in its exterior or encloses Y and has R in its exterior. And since, by hypothesis, C is a maximal cyclic curve of M, it is readily seen that J cannot enclose Y. Hence J encloses R and has Y in its exterior. Now since C is a maximal cyclic curve of M, it follows that J can have at most one point Q in common with C. Then since $(J + Q) - Q$ is a connected subset of $M - C$, which evidently is within C, therefore it belongs to some maximal connected subset N of $M - C$. Now N cannot belong to H, for since X belongs to the segment AB of the arc AB, which contains no point of H, evidently X, and hence also R, must belong to the unbounded complementary domain of $H + P$. Now N has exactly one limit point Q in C, and Q is different from P. But since the segment AX of the arc AB belongs to R and contains no point whatever of M, the connected point set $R + H + P$ contains no point of N but contains a limit point of the exterior of C. Therefore R belongs to the unbounded complementary domain of $N + Q$. This is contrary to the fact that R is within J, a simple closed curve belonging to $N + Q$. Thus the supposition that C is not a part of the boundary of R leads to a contradiction.

Now by a suitable inversion of the plane, the exterior of C can be thrown into the interior of the image of C. And by the same argument as given above it is shown that there exists in E, the exterior of C, a complementary domain D of M whose boundary contains C. Therefore C is the common part of the boundaries of the two complementary domains R and of M.

THEOREM 26. *The boundary of every complementary domain of a maximal cyclic curve of a continuous curve M belongs to the boundary of some single complementary domain of M.*

* *Loc. cit.*

Proof. By Theorem 10 of C. C., the boundary of every complementary domain of a maximal cyclic curve of M is a simple closed curve, and by the same method as used in the proof of Theorem 25 it is shown that there exists a complementary domain of M whose boundary contains this simple closed curve.

THEOREM 27. *If the continuous curve M is not the boundary of any domain, then in order that every proper subcontinuum of M should be the boundary of a domain it is necessary and sufficient that M should be the sum of three arcs joining two points A and B of M and having no other points common to any two of them.*

Proof. The condition is evidently sufficient. I shall show that it is necessary. In my thesis * I proved the following theorem. THEOREM A. *In order that the continuous curve M should be the boundary of a connected domain it is necessary and sufficient that if J is any simple closed curve in M, then* (1) *M is a subset either of $J + I$ or of $J + E$, where I and E respectively denote the interior and exterior of J, and* (2) *if A and B are any two points of J, then $M - (A + B)$* is not connected. Now M must satisfy condition (1) of Theorem A. For suppose, on the contrary, that M contains a simple closed curve J such that I contains a point X of M and E contains a point Y of M. Then M contains an arc XY, and the arc XY contains a proper subarc t containing points of both I and E. But the continuum $t + J$ is a proper subcontinuum of M and hence, by hypothesis, is the boundary of some domain. This is contrary to Theorem A. It follows that M satisfies condition (1) of Theorem A.

Now since, by hypothesis, M is not the boundary of any domain and since M satisfies condition (1) of Theorem A, it follows that M cannot satisfy condition (2) of Theorem A. Hence there exists a simple closed curve J in M and two points X and Y of J such that $M - (X + Y)$ is connected. Let A_0 be a point on one of the segments of J from X to Y, and B_0 a point on the other. Then by a theorem of R. L. Moore's,† $M - (X + Y)$ contains an arc t from A_0 to B_0. There exist points A and B on t which belong to the segments XA_0Y and XB_0Y respectively of J and such that the segment AB of t lies wholly either within J or without J. Let Z be a point on the segment AB of t. Then the three arcs AXB, AYB, and AZB all belong to M, and every two of them have in common exactly the points A and B. That $AXB + AYB + AZB = M$ follows immediately now, using the fact that M satisfies condition (1) of Theorem A. This completes the proof of Theorem 27.

* "Concerning Continua in the Plane," *loc cit.*, Theorem 6.

† "Concerning Continuous Curves in the Plane," *loc. cit.*, Theorem 1.

6. *Properties of a Continuous Curve as determined by its Maximal Cyclic Curves.* The nature of a continuous curve is determined to a large extent by the nature of its maximal cyclic curves. A considerable number of propositions are true about a continuous curve M provided the same propositions are true for all or for some of the maximal cyclic curves of M and vice versa. In this section some propositions of this sort will be given, and in some cases merely stated without proof, since the proofs present no especial difficulty.

THEOREM 28. *In order that every subcontinuum of a continuous curve M should be a continuous curve it is necessary and sufficient that every subcontinuum of each maximal cyclic curve of M should be a continuous curve.*

THEOREM 29. *In order that a continuous curve M should separate the plane it is necessary and sufficient that some maximal cyclic curve of M should separate the plane.* (See Theorem 24.)

THEOREM 30. *If H is any connected subset of a continuous curve M and C is any maximal cyclic curve of M, then $H \cdot C$ is either vacuous or connected.*

Proof. Suppose, on the contrary, that $H \cdot C = H_1 + H_2$, where H_1 and H_2 are mutually separated point sets. Now for each point X of H which belongs to $M - C$, the maximal connected subset M_x of $M - C$ which contains X has exactly one limit point P_x which belongs to C, and P_x must belong either to H_1 or to H_2. Let K_1 denote the set of all those points X of H in $M - C$ such that the corresponding point P_x belongs to H_1, and let K_2 be the set of all such points such that P_x belongs to H_2. Then clearly $H = H_1 + H_2 + K_1 + K_2$.

Now H_1 contains no limit point of K_2. For suppose it does contain a limit point P of K_2. Then since no subset of K_2 every point of which belongs to some single maximal connected subset of $M - C$ can have a limit point in H_1, it follows that K_2 contains an infinite subset D which has P for its sequential limit point and such that no two points of D lie together in the same maximal connected subset of $M - C$. Let G denote the collection of maximal connected subsets of $M - C$ each of which contains a point of D. Since P is a limit point of D, then P belongs to the limiting set of the collection G. Since P is not a limit point of H_2, there exists a number ϵ such that every point of H_2 is at a distance greater than ϵ from P. But now each element of G has a limit point in H_2, and therefore the limiting set of G has a point whose distance from P is $\geq \epsilon$. It follows that infinitely many elements of G must be of diameter $> \epsilon/2$. This is contrary to a theorem of

W. L. Ayres.* Thus the supposition that H_1 contains a limit point of K_2 leads to a contradiction.

Now since H_1 belongs to C and K_2 belongs to $M - C$, it is clear that no point of K_2 can be a limit point of H_1. Hence H_1 and K_2 are mutually separated point sets. A similar argument shows that H_2 and K_1 are mutually separated. Hence, as H_1 and H_2, by supposition, are mutually separated, and clearly K_1 and K_2 are mutually separated, it follows that $(H_1 + K_1)$ and $(H_2 + K_2)$ are mutually separated point sets. But the sum of these two sets is H, and H, by hypothesis, is connected. Thus the supposition that $H \cdot C$ is neither vacuous nor connected leads to a contradiction, and the theorem is established.

THEOREM 31. *If the subcontinuum C of a continuous curve M has no cut point, then in order that C should be a maximal cyclic curve of M it is necessary and sufficient that if H is any connected subset of M, then $H \cdot C$ is either vacuous or connected.*

Proof. Theorem 30 shows that the condition is necessary. To show that it is sufficient, let us first show that C is a continuous curve. Suppose the contrary is true. Then there exists a point P of C and a positive number ϵ such that no matter what number δ we choose, there are points of C whose distance from P is less than δ but which lie in no connected subset of C containing P and of diameter less than ϵ. Now since M is connected im kleinen at P, there exists a positive number δ_ϵ such that every point of M whose distance from P is $< \delta_\epsilon$ can be joined to P by an arc † of M of diameter less than ϵ. Let X be a point of C whose distance from P is $< \delta_\epsilon$ and which cannot be joined to P by any connected subset of C of diameter less than ϵ. There exists in M an arc XP of diameter less than ϵ. The arc XP is not a subset of C. Hence there exists an interior point O of XP not belonging to C. On the arcs OX and OP of XP, in the order from O to X and from O to P respectively, let A and B denote the first points belonging to C. Then the arc AOB of XP is a connected subset of M having in common with C only the point set $A + B$, which clearly is not connected. This is contrary to hypothesis, and it follows that C is a continuous curve.

Since C has no cut point, then by Theorem 1 of C. C., C is cyclicly connected. Now suppose, contrary to this theorem, that C is not a maximal cyclic curve of M. Then by Theorem 2 of C. C., there exists a connected

---

* W. L. Ayres, *loc. cit.* The special case of Ayres' theorem here used holds in n dimensions.

† Cf. J. R. Kline, "Concerning Approachability of Simple Closed and Open Curves," *Transactions of the American Mathematical Society*, Vol. 21 (1920), footnote to p. 453.

subset N of $M - C$ which has at least two limit points A and B which belong to C. This is impossible, because the connected subset $N + A + B$ of M has only the point set $A + B$ in common with C, and $A + B$ is not connected. Thus the supposition that C is not a maximal cyclic curve of M leads to a contradiction.

It is interesting to note that with the aid of Theorem 30 the proof of Theorem 1 can be considerably shortened.

THEOREM 32. *If H is a connected subset of a continuous curve M such that the subset of H contained in each maximal cyclic curve of M is arcwise connected, then H itself is arcwise connected.*

Proof. Let A and B be any two points of H, let t be any arc in M from A to B, and let K denote the set of all those points of M which separate A and B in M. Clearly K belongs to both T and H. By a theorem of the author's,* $K + A + B$ is a closed set of points. Hence $t - (K + A + B)$ is the sum of a countable number of non-overlapping segments $S_1, S_2, S_3, \cdots$ For each positive integer i, let A_i and B_i denote the endpoints of S_i. For each i, S_i determines † a maximal cyclic curve C_i of M which contains S_i. Now for each i, A_i and B_i belong to $K + A + B$, and hence belong also to H and to C_i. Then since by hypothesis $H \cdot C_i$ is arcwise connected and contains both A_i and B_i, therefore it contains an arc t_i from A_i to B_i Now let

$$T = K + A + B + \sum_{i = 1, 2, 3, \cdots} t_i .$$

Then T is a subset of H, and since for no two integers k and j can t_k and t_j have an interior point of either in common, and since for no positive number ϵ does there exist more than a finite number of maximal cyclic curves of M of diameter greater than ϵ, it is easy to see that T is a simple continuous arc from A to B. Therefore H is arcwise connected.

THEOREM 33. *In order that every connected subset of a continuous curve M should be arcwise connected it is necessary and sufficient that every connected subset of each maximal cyclic curve of M should be arcwise connected.*

Theorem 33 follows at once from Theorems 32 and 31.

THEOREM 34. *If every maximal cyclic curve of a continuous curve M is a simple closed curve, then every connected subset of M is arcwise connected.*

Theorem 34 is a corollary to Theorem 33.

It is to be noted that Theorem 34 includes the following important

* G. T. Whyburn, "Some Properties of Continuous Curves," *loc. cit.*, Theorem 1.

† *Loc. cit.*, Theorem 2.

theorem of R. L. Wilder's *: every connected subset of the boundary of a complementary domain of a continuous curve is arcwise connected. It is readily seen with the aid of Theorem 25 that every maximal cyclic curve of the boundary B of a complementary domain of a continuous curve is a simple closed curve. Hence, by Theorem 34, every connected subset of B is arcwise connected.

In connection with Theorems 25 and 34, the following proposition is interesting because of the light it throws upon the nature of a continuous curve every maximal cyclic curve of which is a simple closed curve.

THEOREM 35. *If every maximal cyclic curve of a continuous curve M is a simple closed curve, then M is the sum of its endpoints plus the boundaries of all of its complementary domains.*

Proof. Suppose, on the contrary, that M contains a point P which neither is an endpoint of M nor belongs to the boundary of any complementary domain of M. Since P is not an endpoint of M, by the definition of an endpoint, there exists an arc APB in M having P as an interior point. Let d be the smaller of the two numbers d_1 and d_2, where d_1 is $\frac{1}{2}$ the distance from A to P and d_2 is $\frac{1}{2}$ the distance from B to P. Then since P belongs to the boundary of no complementary domain of M, by a theorem of mine,† there exists in M a simple closed curve J which encloses P and is of diameter less than d. Clearly J does not enclose either A or B. In the order from P to A and from P to B respectively on APB, let X and Y denote the first points belonging to J. Then J is not a maximal cyclic curve of M, because the segment XPY of the arc APB belongs to $M - J$ and has two limit points in J. This is contrary to Theorem 2 of C. C. Hence J is not a maximal cyclic curve of M. But J belongs to some maximal cyclic curve C of M. Therefore C is not a simple closed curve, contrary to hypothesis. Thus the supposition that Theorem 35 is false leads to a contradiction.

In conclusion I wish to point out that the above theory of the structure of a continuous curve as being composed of its cyclic elements appears much more perfect if we restrict the types of continuous curves considered to any one of the following three types: (1) continuous curves no two cyclic elements of which have a common point, (2) continuous curves which are the boundary of some domain, or (3) continuous curves which do not separate the plane (see Theorem 24).

UNIVERSITY OF TEXAS,
AUSTIN, TEXAS.

* R. L. Wilder, "Concerning Continuous Curves," *loc. cit.*, Theorem 20.

† "Concerning Continua in the Plane," *loc. cit.*, Theorem 7.

Compound Statements on Four Classes.

By Frances E. Baker.

1. *Introduction.* W. K. Clifford * first enumerated the compound statements on four classes, applying an intuitive method which allowed no check. Cayley † carried a check as far as three-fold statements. Up to this stage, the point of view of simple combinations was sufficient, but further progress demanded the use of a theorem of groups, viz., that if an object is invariant under a subgroup of order N, and the total group has order M, M/N is the number of conjugates. In the present paper this check is carried out completely, showing that in several cases Clifford was in error.

The order M of the group is here $384 = 4! \times 2^4$, representing the number of permutations of the four classes multiplied by the group of logical operations whose primitive elements are negations. Certain theorems about possible subgroups are proved.

A special notation proves convenient in which $[\alpha]$ means logical multiplication, or denial of the class A; and the usual signs, as $(\alpha\beta)$, mean interchange of classes. The rules for transposition of the square and round brackets are relatively simple. In the case of the eight-fold classes, the material to be worked upon is obtained by a special enumeration which gives duplicates but is inclusive.

2. *Notation.* To facilitate the counting of distances, and to serve as a basis for the subsequent group technique, a new notation has been introduced, namely:

The identical element	$I \sim ABCD = 0$	(no $ABCD$ exists)
	$\alpha \sim aBCD = 0$	(no $aBCD$ exists)
	$\beta \sim AbCD = 0$	(no $AbCD$ exists)
	$\gamma \sim ABcD = 0$	(no $ABcD$ exists)
	$\delta \sim ABCd = 0$	(no $ABCd$ exists)

* Clifford, W. K., *Memoirs of the Literary and Philosophical Society of Manchester*, Vol. 16, No. 7, pp. 88-101. Also, *Mathematical Papers*, by W. K. Clifford, pp. 1-16; and *Lectures and Essays*, by W. K. Clifford, Vol. 2, pp. 89-105. For the origin of the problem, see also: Jevons, W. Stanley, *Proceedings of the Manchester Philosophical Society*, Vol. 6, pp. 65-68, and *Memoirs*, Third Series, Vol. 5, pp. 119-130; *The Principles of Science*, Vol. 1, pp. 154-164. [New Edition, pp. 134-143.]

† Cayley, A., *Memoirs of the Literary and Philosophical Society of Manchester*, Vol. 16, p. 113; *Collected Papers*, Vol. 10, p. 243.

Consequently: $\alpha\beta \sim abCD = 0$
$\alpha\beta\gamma \sim abcD = 0$
$\alpha\beta\gamma\delta \sim abcd = 0$ etc.

Economy of letters is an advantage gained. For example, $BCD = 0$ in the old notation representing the two-fold statement $ABCD = 0$, $aBCD = 0$, may now be expressed simply as I, α. Similarly, $AC = 0$, standing for $ABCD = 0$, $AbCD = 0$, $ABCd = 0$, $AbCd = 0$, becomes I, β, δ, $\beta\delta$. It may be noted that statements denying entirely a particular class, such as the eight-fold statement $I\ \alpha\ \beta\ \gamma\ \alpha\beta\ \alpha\gamma\ \beta\gamma\ \alpha\beta\gamma$, are inconsistent according to Jevons,* but are included by Clifford.

If we now consider I as the origin, the 4 proximates become $\alpha\ \beta\ \gamma\ \delta$; the 6 mediates become $\alpha\beta\ \alpha\gamma\ \alpha\delta\ \beta\gamma\ \beta\delta\ \gamma\delta$; the 4 ultimates become $\alpha\beta\gamma\ \alpha\beta\delta\ \alpha\gamma\delta\ \beta\gamma\delta$; and the obverse to I becomes $\alpha\beta\gamma\delta$. Hence there are altogether **16 marks:**

$$I\ \alpha\ \beta\ \gamma\ \delta\ \alpha\beta\ \alpha\gamma\ \alpha\delta\ \beta\gamma\ \beta\delta\ \gamma\delta\ \alpha\beta\gamma\ \alpha\beta\delta\ \alpha\gamma\delta\ \beta\gamma\delta\ \alpha\beta\gamma\delta.$$

This set of 16 objects is invariant as a whole under:

(1). Multiplication by any member of itself. Let [ ] be the symbol for multiplication, so that $[\alpha\beta\delta]$ means multiply by $\alpha\beta\delta$. There are 16 multipliers, in which $\alpha\ \beta\ \gamma\ \delta$ are commutative. Multiplication is taken modulo 2 with respect to exponents. Thus $[\alpha^\pi\beta^\kappa\gamma^\rho\delta^\sigma]$ represents a general multiplier, where $\pi, \kappa, \rho, \sigma = (0, 1)$.

Ex. $[\alpha^0\beta^0\gamma^1\delta^1][\alpha^0\beta^1\gamma^1\delta^0] = [\gamma\delta][\beta\gamma] = I \cdot \beta^1 \cdot I \cdot \delta^1 = [\beta\delta]$.

The permutations of the set $I\ \alpha\ \beta\ \gamma \cdot \cdot \cdot \alpha\beta\gamma\delta$ effected by [ ] all change I.

(2). 24 substitutions, viz., the $G^4{}_{24}$. The permutations of the set $I\ \alpha\ \beta\ \gamma \cdot \cdot \cdot \alpha\beta\gamma\delta$ effected by ( ) do not change I.

3. *The Order of the Group of Permutations.* To prove that the order of the group of permutations of the set $I\ \alpha\ \beta\ \gamma \cdot \cdot \cdot \alpha\beta\gamma\delta$ is 16×24:

Every operation is reducible either to [] () or to () [] or to a succession of [] () [] () $\cdot : \cdot$. As a general element of the set take $E = \{\alpha^p\beta^q\gamma^r\delta^s\}$ and as elements of multiplication and substitution, respectively, take $M_1 = [\alpha^\pi\beta^\kappa\gamma^\rho\delta^\sigma]$ and $S_1 = (\alpha\beta)$ where $p, q, r, s, \pi, \kappa, \rho, \sigma = (0, 1)$. Let {} [] () mean first multiply into the operand, then permute; and let {} () [] mean first permute, then multiply into the result.

---

* *Loc. cit.*

Examples:

$$(1). \quad \left\{ \begin{matrix} \alpha^p\beta^q\gamma^r\delta^s \\ E \end{matrix} \right\} \underset{M_1}{[\alpha^\pi\beta^\kappa\gamma^\rho\delta^\sigma]} \underset{S_1}{(\alpha\beta)} = \left\{ \begin{matrix} \alpha^{p+\pi}\beta^{q+\kappa}\gamma^{r+\rho}\delta^{s+\sigma} \\ E_1 \end{matrix} \right\} \underset{S_1}{(\alpha\beta)}$$

$$= \left\{ \begin{matrix} \alpha^{q+\kappa}\beta^{p+\pi}\gamma^{r+\rho}\delta^{s+\sigma} \\ E_2 \end{matrix} \right\}$$

Since we can write:

$$E_2 = \left\{ \begin{matrix} \alpha^p\beta^q\gamma^r\delta^s \\ E \end{matrix} \right\} \underset{S_1}{(\alpha\beta)} \underset{M_2}{[\alpha^\pi\beta^\kappa\gamma^\rho\delta^\sigma]}$$

it is seen that $EM_1S_1 = ES_1M_2$. Treating the multiplication symbols as substitution symbols:

$$ES_1^{-1}M_1S_1 = ES_1^{-1}S_1M_2$$
$$\therefore\ S_1^{-1}M_1S_1 = M_2$$
$$\text{i. e. } (\alpha\beta)^{-1}M_1(\alpha\beta) = M_2$$

(2). Likewise

$$[M_1](\alpha\beta)(\alpha\gamma) = (\alpha\beta)[M_2](\alpha\gamma)$$
$$= (\alpha\beta)(\alpha\gamma)[M_3]$$
$$\text{where } [M_2] = (\alpha\beta)^{-1}[M_1](\alpha\beta)$$
$$[M_3] = (\alpha\gamma)^{-1}(\alpha\beta)^{-1}[M_1](\alpha\beta)(\alpha\gamma)$$
$$= (\alpha\beta\gamma)^{-1}[M_1](\alpha\beta\gamma)$$

(3). Similarly

$$[M_1](\alpha\beta)(\alpha\gamma)(\alpha\delta) = (\alpha\beta)(\alpha\gamma)(\alpha\delta)[M_2]$$
$$[M_1](\alpha\beta\gamma\delta) = (\alpha\beta\gamma\delta)[M_2]$$
$$\text{where } [M_2] = (\alpha\beta\gamma\delta)^{-1}[M_1](\alpha\beta\gamma\delta).$$

It is evident that when the substitution is moved from the right of the multiplier to the left, the new multiplier is the transform of the old by the substitution. Proceeding in the same way, we find upon reversing the order of multiplier and substitution that:

$$(4). \quad (\alpha\beta)[M_1] = [M_2](\alpha\beta)$$
$$[M_2] = \{(\alpha\beta)^{-1}\}^{-1}[M_2](\alpha\beta)^{-1}$$

$$(5). \quad (\alpha\beta)(\alpha\gamma)[M_1] = [M_2](\alpha\beta)(\alpha\gamma)$$
$$[M_2] = \{(\alpha\beta\gamma)^{-1}\}^{-1}[M_1](\alpha\beta\gamma)^{-1}$$

$$(6). \quad (\alpha\beta)(\alpha\gamma)(\alpha\delta)[M_1] = [M_2](\alpha\beta)(\alpha\gamma)(\alpha\delta)$$
$$[M_2] = \{(\alpha\beta\gamma\delta)^{-1}\}^{-1}[M_1](\alpha\beta\gamma\delta)^{-1}.$$

Obviously, when the substitution is moved from the left of the multiplier to the right, the new multiplier is the transform of the old by the inverse of the substitution.

It is to be observed that similar multipliers and substitutions permute; also multipliers and substitutions neither of which contain any marks of the other.

Ex.
$$[\alpha\beta](\alpha\beta) = (\alpha\beta)[\alpha\beta]$$
$$(\alpha\beta\gamma\delta)[\alpha\beta\gamma\delta] = [\alpha\beta\gamma\delta](\alpha\beta\gamma\delta)$$
$$(\alpha\beta)[\gamma\delta] = [\gamma\delta](\alpha\beta)$$

Since every substitution can be expressed as the product of transpositions, the laws for the right-hand and left-hand transposition of the substitution symbol hold for all substitutions. Consequently the group of permutations contains only operations reducible to the form [] (). Therefore its order is 16×24, since these operations are distinct.

4. *Discussion of Subgroups.* As the multiplication laws show that the 16 multipliers do not affect the 24 substitutions, the possible orders of subgroups of the $G^4{}_{16\times 24}$ are those of the orders of the subgroups of the $G^4{}_{24}$, which may be multiplied by 2^k where $k = 0, 1, 2, 3, 4$. The subgroups are therefore of orders: 1, 2, 4, 6, 8, 16, 24, 32, 48, 64, 96, 192, 384.

G_{12} does not occur, for a two-valued function of α, β, γ, δ, necessarily contains $(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\beta-\gamma)(\beta-\delta)(\gamma-\delta)$, which of course does not occur in this mere collection.

C_4 does not occur, because if C_4 leaves a collection invariant then if α occurs, α, β, γ, δ occur; if $\alpha\beta\gamma$ occurs, $\beta\gamma\delta$, $\gamma\delta\alpha$, $\delta\alpha\beta$, occur; if $\alpha\beta$ occurs, $\beta\gamma$, $\gamma\delta$, $\delta\alpha$ are present. At most only $\alpha\gamma$, $\beta\delta$ are absent, and these are invariant under G_8.

5. *Determination and Description of Types.* In the main, Clifford's work has been followed, the types overlooked by him in the six- and seven-fold statements having been found by trial. However, the number of mistakes in the eight-fold statements makes necessary for their detection an independent device, which gives some duplicates but includes all the possible types.

(1). The determination of the eight-fold set is accomplished in the following manner:— If one letter, say δ, not occurring oftener than any other be abstracted from an eight-fold type, the residue may be called a base. It represents a set of eight points of a cube, and may be the whole set $I\ \alpha\ \beta\ \gamma\ \alpha\beta\ \alpha\gamma\ \beta\gamma\ \alpha\beta\gamma$ or may have some duplicates and lack the same number. Not more than two can be alike, otherwise in the original set two would

have agreed. The types of duplication and omission which are distinct may be classified by geometrical intuition on the cube, and verified by counting the number of conjugates and finding the total.

Ex. If two are omitted and two duplicated, the duplicated may be an edge, a face diagonal or a cube diagonal, and so for the omitted, but they can have no common members.

We get such symbols as $[I, \alpha](\gamma, \alpha\gamma)$, read "Double I, α, omit γ, $\alpha\gamma$." These are opposite edges of the same face. The symbol is invariant under $[\alpha]$, $[\gamma](12)$, i. e. under multiplication by α and under multiplication by γ followed by interchange of parts. Since the whole group on three classes is of order $2^3 \cdot 3! = 48$, and this subgroup is of order 4, there are 12 conjugates to it. Proceeding in this way until we have in all ${}_8C_2 \cdot {}_6C_2/2 = 210$ conjugates, we have a check. Similarly are treated the cases in which one is doubled, one omitted; three are doubled, three omitted; four are doubled, four omitted.

The set of possible bases is then set out, 32 in all. There is 1 base when none are doubled; there are 3 when one is doubled; 12 when two are doubled; 10 when three are doubled; 6 when four are doubled. Each base is now filled with δ's in all possible ways so that the δ does not occur oftener than any other letter. Repetitions which are ineffective because of a group invariance of the base are omitted. Duplications will arise if the number of δ's is equal to that of the γ's, for instance. Actually, out of 108 types found by this method, 74 are distinct.

(2). Throughout, the types have been named with reference to their sets of distances as a preliminary distinction. Taking the distance in a type of every mark from its successive neighbors, the number of times each distance occurs is noted. The five-fold statement $I\ \alpha\beta\gamma\delta\ \alpha\ \beta\gamma\ \alpha\delta$, for example, has distances 4122 : 322 : 31 : 4. The fact that 1 occurs twice, 2 occurs 4 times, 3 twice, 4 twice, is expressed as 2422. Likewise the eight-fold statement $I\ \alpha\ \beta\ \delta\ \alpha\beta\ \alpha\gamma\ \beta\delta\ \gamma\delta$ with distances 1112222 : 221133 : 21313 : 3311 : 224 : 42 : 2 is called 9, 11, 62.

Appearance of more than one type with the same set of distances necessitates further differentiation. As the above eight-fold type contains 3 α's, 3 β's, 2 γ's, 3 δ's, it may be designated a member of the 3323 set, indicated in the tables under "letters." Moreover, the three marks containing α form a triangle with distances 112; the three containing β form a triangle with distances 112; and the three containing δ form a triangle with distances 112. Because two other kinds of triangle are also possible, namely those with dis-

tances 123, and 222, the three kinds may be conveniently distinguished as numbers 1, 2 and 3. Since the marks containing γ are at distance 2, the above type 9, 11, 62 bears a further classification, 1121, under "basis."

When an eight-fold type exhibits a letter occurring four times, the four marks represent four corners of a cube. These sets are classified in the following order:

1.	$I\ \alpha\ \beta\ \gamma$	111222
2.	$I\ \alpha\ \beta\ \alpha\beta$	112211
3.	$I\ \alpha\ \beta\ \alpha\beta\gamma$	113222
4.	$I\ \alpha\ \alpha\beta\ \alpha\beta\gamma$	123121
5.	$I\ \alpha\beta\ \beta\gamma\ \alpha\gamma$	222222
6.	$I\ \alpha\ \beta\gamma\ \alpha\beta\gamma$	123321

$I\ \beta\ \gamma\ \alpha\beta\ \alpha\delta\ \beta\delta\ \alpha\beta\gamma\ \alpha\gamma\delta$ can now be described thus:

Letters	*Distance*	*Basis*
4433	6, 11, 10, 1	6432

It allows a 2 group, with generators I, $[\alpha\beta](\alpha\beta)(\gamma\delta)$ and 192 conjugates.

(3). Should an n-fold type display a letter in more than $E\{n/2\}$ marks, it is impossible to determine the kinds of triangles and quadrilaterals until the type has been reduced to a minimum form in which for two- and three-fold types no letter appears more than once; for four- and five-fold types no letter appears more than twice; for six- and seven-fold types no letter appears more than three times; for eight-fold types no letter appears more than four times. Obviously this reduction can be accomplished by multiplication by the offending letter or letters. If the multiplier be not a member of the type, the identity will be lost. In the tables, the types are set out in minimum form.

Note that, in the tables, (all) means all the permutations on the letters, whereas [all] means multiplication by all the members, of the type form listed.

6. *Errors in Clifford's Work.* This table compares the present results with those of Clifford.

Type	*Clifford*	*Total*	*Corrected*	*Total*	*No. of Conjugates*
One-fold	1	1	1	1	$16 = {}_{16}C_1$
Two-fold	4	4	4	4	$120 = {}_{16}C_2$
Three-fold	6	6	6	6	$560 = {}_{16}C_3$
Four-fold					
Pure	12		12		
1 pair obverses....	5		5		
2 pair obverses....	2		2		
		19		19	$1820 = {}_{16}C_4$
Five-fold					
Pure	12		12		
1 pair obverses....	12		12		
2 pair obverses....	3		3		
		27		27	$4368 = {}_{16}C_5$
Six-fold					
Pure	12		14		
1 pair obverses....	22		22		
2 pair obverses....	11		12		
3 pair obverses....	2		2		
		47		50	$8008 = {}_{16}C_6$
Seven-fold					
Pure	8		8		
1 pair obverses....	24		23		
2 pair obverses....	19		20		
3 pair obverses....	4		5		
		55		56	$11,440 = {}_{16}C_7$
Total < eight-fold..		159		163	26,332
Complementary		159		163	26,332
Eight-fold					
Pure	7		7		
1 pair obverses....	14		17		
2 pair obverses....	41		33		
3 pair obverses....	12		13		
4 pair obverses....	4		4		
		78		74	$12,870 = {}_{16}C_8$
Grand Total.....		396		400	$65,534 = 2^{24}$

The last column of statements forms the final check of the work.

7. *Tables of Statements.* There is of course only one type of simple statement, which may be represented as I, having a group of order 24, since it is susceptible to all interchanges, with 16 conjugates.

Four Classes; Two-fold Statements.

Letters	Distance	Type	Generators	Group Order	Conjugates
1111	0001	$I\ \alpha\beta\gamma\delta$	$(\alpha\beta)$; $(\alpha\gamma)$ $(\alpha\delta)$; $[\alpha\beta\gamma\delta]$	48	8
1110	0010	$I\ \alpha\beta\gamma$	$(\alpha\beta)$; $(\alpha\gamma)$; $[\alpha\beta\gamma]$	12	32
1100	0100	$I\ \alpha\beta$	$(\alpha\beta)$; $(\gamma\delta)$; $[\alpha\beta]$	8	48
1000	1000	$I\ \alpha$	$(\beta\gamma)$; $(\beta\delta)$; $[\alpha]$	12	32
					120

Four Classes; Three-fold Statements

Letters	Distance	Type	Generators	Group Order	Conjugates
1111	0120	$\alpha\ \beta\ \gamma\delta$	$(\alpha\beta)$; $(\gamma\delta)$	4	96
	0201	$I\ \alpha\beta\ \gamma\delta$	$(\alpha\beta)$; $(\alpha\gamma\beta\delta)$	8	48
	1011	$I\ \alpha\ \beta\gamma\delta$	$(\beta\gamma)$; $(\beta\delta)$	6	64
1110	0300	$\alpha\ \beta\ \gamma$	$(\alpha\beta)$; $(\alpha\gamma)$	6	64
	1110	$I\ \alpha\ \beta\gamma$	$(\beta\gamma)$	2	192
1100	2100	$I\ \alpha\ \beta$	$(\alpha\beta)$; $(\gamma\delta)$	4	96
					560

Four Classes; Four-fold Statements.

Letters	Distance	Basis	Type	Generators	Group Order	Conjugates
2222	0240		$I\ \alpha\beta\ \alpha\gamma\delta\ \beta\gamma\delta$	$(\alpha\beta)$; $(\gamma\delta)$; [all]	16	24
	0402		$I\ \alpha\beta\ \gamma\delta\ \alpha\beta\gamma\delta$	$(\alpha\beta)$; $(\alpha\beta\gamma\delta)$; [all]	32	12
	2022		$I\ \alpha\ \beta\gamma\delta\ \alpha\beta\gamma\delta$	$(\beta\gamma)$; $(\beta\delta)$; [all]	24	16
2221	0330		$I\ \alpha\beta\ \beta\gamma\ \alpha\gamma\delta$	$(\alpha\gamma)$; $[\alpha\beta]$ $(\alpha\beta)$	6	64
	1221		$I\ \alpha\ \beta\gamma\ \alpha\beta\gamma\delta$	$(\beta\gamma)$	2	192
2220	0600		$I\ \alpha\beta\ \alpha\gamma\ \beta\gamma$	$(\alpha\beta)$; $(\alpha\gamma)$; [all]	24	16
	2220		$I\ \alpha\ \beta\gamma\ \alpha\beta\gamma$	$(\beta\gamma)$; [all]	8	48
2211	0501		$I\ \alpha\beta\ \alpha\delta\ \beta\gamma$	$(\alpha\beta)$; $(\gamma\delta)$; $[\alpha\beta](\alpha\beta)$	4	96
	1230		$I\ \alpha\ \beta\gamma\ \alpha\beta\delta$	$[\alpha](\gamma\delta)$	2	192
	2121	33	$I\ \alpha\ \beta\ \alpha\beta\gamma\delta$	$(\alpha\beta)$; $(\gamma\delta)$	4	96
	2121	11	$I\ \gamma\ \alpha\beta\ \alpha\beta\delta$	$(\alpha\beta)$; $[\alpha\beta](\gamma\delta)$	4	96
2210	2310		$I\ \alpha\ \beta\ \alpha\beta\gamma$	$(\alpha\beta)$	2	192
2200	4200		$I\ \alpha\ \beta\ \alpha\beta$	$(\alpha\beta)$; $(\gamma\delta)$; [all]	16	24
2111	1320		$I\ \delta\ \alpha\beta\ \alpha\gamma$	$(\beta\gamma)$	2	192
	2211		$I\ \alpha\ \beta\ \alpha\gamma\delta$	$(\gamma\delta)$	2	192
2110	3210		$I\ \alpha\ \beta\ \alpha\gamma$	$[\alpha](\beta\gamma)$	2	192
1111	0600		$\alpha\ \beta\ \gamma\ \delta$	$(\alpha\beta)$; $(\alpha\gamma)$; $(\alpha\delta)$	24	16
	2220		$I\ \alpha\ \delta\ \beta\gamma$	$(\alpha\delta)$; $(\beta\gamma)$	4	96
1110	3300		$I\ \alpha\ \beta\ \gamma$	$(\alpha\beta)$; $(\alpha\gamma)$	6	64
						1820

Four Classes; Five-fold Statements.

Letters	Distance	Basis	Type					Generators	Group Order	Conjugates
2222	0640		α	β	γ	δ	$\alpha\beta\gamma\delta$	$(\alpha\beta)$; $(\alpha\gamma)$; $(\alpha\delta)$	24	16
	0802		I	$\alpha\beta$	$\alpha\gamma$	$\beta\delta$	$\gamma\delta$	$(\alpha\delta)$; $(\alpha\beta)(\gamma\delta)$	8	48
	1450		I	α	$\beta\gamma$	$\gamma\delta$	$\alpha\beta\gamma$	$(\beta\gamma)$	2	192
	1531		α	γ	δ	$\beta\delta$	$\alpha\beta\gamma$	$(\alpha\gamma)$	2	192
	2341	1122	α	β	$\alpha\gamma$	$\beta\delta$	$\gamma\delta$	$(\alpha\beta)(\gamma\delta)$	2	192
	2341	3232	I	α	γ	$\beta\delta$	$\alpha\beta\gamma\delta$	$(\alpha\gamma)$; $(\beta\delta)$	4	96
	2422		I	α	β	$\alpha\gamma\delta$	$\beta\gamma\delta$	$(\alpha\beta)$; $(\gamma\delta)$	4	96
	3232		I	α	$\alpha\beta$	$\gamma\delta$	$\beta\gamma\delta$	$(\gamma\delta)$	2	192
2221	0901		α	β	γ	δ	$\alpha\beta\gamma$	$(\alpha\beta)$; $(\alpha\gamma)$	6	64
	1630		I	δ	$\alpha\beta$	$\alpha\gamma$	$\beta\gamma$	$(\alpha\beta)$; $(\alpha\gamma)$	6	64
	2440	223	I	α	β	$\gamma\delta$	$\alpha\beta\gamma$	$(\alpha\beta)$	2	192
	2440	112	α	β	δ	$\alpha\gamma$	$\beta\gamma$	$(\alpha\beta)$	2	192
	2521		I	β	$\alpha\gamma$	$\alpha\delta$	$\beta\gamma$		1	384
	3331	333	I	α	β	γ	$\alpha\beta\gamma\delta$	$(\alpha\beta)$; $(\alpha\gamma)$	6	64
	3331	111	α	β	γ	$\alpha\delta$	$\beta\gamma$	$(\beta\gamma)$	2	192
	3331	211	I	α	δ	$\beta\gamma$	$\alpha\beta\gamma$	$(\beta\gamma)$	2	192
	3331	213	I	α	β	$\beta\gamma$	$\alpha\gamma\delta$		1	384
2220	3610		I	α	β	γ	$\alpha\beta\gamma$	$(\alpha\beta)$; $(\alpha\gamma)$	6	64
	4420		I	α	β	$\alpha\gamma$	$\beta\gamma$	$(\alpha\beta)$	2	192
2211	2620		α	β	γ	δ	$\alpha\beta$	$(\alpha\beta)$; $(\gamma\delta)$	4	96
	3430		I	β	δ	$\alpha\beta$	$\alpha\gamma$		1	384
	3511		I	α	β	γ	$\alpha\beta\delta$	$(\alpha\beta)$	2	192
	4321	11	I	α	β	$\alpha\delta$	$\beta\gamma$	$(\alpha\beta)(\gamma\delta)$	2	192
	4321	11	I	α	β	$\alpha\beta$	$\gamma\delta$	$(\alpha\beta)$; $(\gamma\delta)$	4	96
2210	5410		I	α	β	γ	$\alpha\beta$	$(\alpha\beta)$	2	192
2111	4420		I	α	β	γ	$\alpha\delta$	$(\beta\gamma)$	2	192
1111	4600		I	α	β	γ	δ	$(\alpha\beta)$; $(\alpha\gamma)$; $(\alpha\delta)$	24	16
										4368

Four Classes; Six-fold Statements.

Letters	Distance	Basis	Type					Generators	Group Order	Conjugates
3333	0, 12, 03		I	$\alpha\gamma$	$\alpha\delta$	$\beta\gamma$	$\beta\delta$	$\alpha\beta\gamma\delta$	$(\alpha\beta)$; $(\gamma\delta)$; $[\alpha\beta\gamma\delta]$; $[\beta\gamma](\beta\gamma)$	48	8
	2661		I	α	$\beta\gamma$	$\gamma\delta$	$\alpha\beta\delta$	$\alpha\beta\gamma\delta$	$(\beta\delta)$; $[\alpha\beta\gamma\delta](\alpha\gamma)$	4	96
	4443		I	α	$\alpha\delta$	$\beta\gamma$	$\beta\gamma\delta$	$\alpha\beta\gamma\delta$	$(\beta\gamma)$; $[\beta\gamma](\alpha\delta)$; $[\alpha\beta\gamma\delta]$	8	48
3332	1941		α	β	γ	δ	$\alpha\beta\gamma$	$\alpha\beta\gamma\delta$	$(\alpha\beta)$; $(\alpha\gamma)$	6	64
	2670		I	α	$\beta\gamma$	$\beta\delta$	$\alpha\beta\gamma$	$\alpha\gamma\delta$	$[\alpha\beta\gamma]$; $[\beta\gamma](\beta\gamma)$	4	96
	2832		α	γ	δ	$\beta\gamma$	$\alpha\beta\gamma$	$\alpha\beta\delta$	$[\alpha\beta](\alpha\beta)$	2	192
	3561		I	α	γ	$\beta\delta$	$\alpha\beta\gamma$	$\alpha\beta\gamma\delta$	$(\alpha\gamma)$	2	192
	4452	1213	I	β	δ	$\alpha\gamma$	$\alpha\beta\gamma$	$\alpha\beta\gamma\delta$	$(\alpha\gamma)$; $[\alpha\beta\gamma]$	4	96
	4452	2221	I	α	γ	$\alpha\beta$	$\beta\gamma\delta$	$\alpha\beta\gamma\delta$	$[\alpha](\beta\gamma)$	2	192
3331	4641	222	I	α	β	γ	$\alpha\beta\gamma$	$\alpha\beta\gamma\delta$	$(\alpha\beta)$; $(\alpha\gamma)$	6	64
	4641	232	I	α	γ	$\alpha\beta$	$\beta\gamma$	$\alpha\beta\gamma\delta$	$(\alpha\gamma)$	2	192
3330	6630		I	α	γ	$\alpha\beta$	$\beta\gamma$	$\alpha\beta\gamma$	$(\alpha\gamma)$; $[\alpha](\beta\gamma)$; $[\alpha\beta\gamma]$	12	32
3322	0, 13, 02		α	β	γ	δ	$\alpha\beta\gamma$	$\alpha\beta\delta$	$(\alpha\beta)$; $(\gamma\delta)$; $[\alpha\beta]$	8	48
	2760	3233	I	β	$\alpha\beta$	$\alpha\gamma$	$\alpha\delta$	$\beta\gamma\delta$	$(\gamma\delta)$; $[\alpha\beta](\alpha\beta)$	4	96

Four Classes; Six-fold Statements—(*Continued*).

Letters	Distance	Basis	Type						Generators	Group Order	Conjugates
	2760	2222	α	β	γ	$\alpha\delta$	$\beta\delta$	$\alpha\beta\gamma$	$(\alpha\beta)$; $[\alpha\beta]$	4	96
	3651	3232	I	β	γ	$\alpha\beta$	$\alpha\delta$	$\alpha\beta\gamma\delta$		1	384
	3651	1212	α	γ	δ	$\alpha\beta$	$\beta\gamma$	$\alpha\beta\delta$		1	384
	4542	2222	I	α	β	$\alpha\beta$	$\gamma\delta$	$\alpha\beta\gamma\delta$	$(\alpha\beta)$; $(\gamma\delta)$; $[\alpha\beta]$	8	48
	4542	2222	I	α	β	$\alpha\delta$	$\beta\gamma$	$\alpha\beta\gamma\delta$	$(\alpha\beta)(\gamma\delta)$	2	192
	4542	2213	I	α	β	δ	$\alpha\beta\gamma$	$\alpha\beta\gamma\delta$	$(\alpha\beta)$	2	192
	4542	2111	I	γ	$\alpha\beta$	$\alpha\gamma$	$\beta\delta$	$\alpha\beta\delta$	$[\alpha\beta](\gamma\delta)$	2	192
	4542	1122	I	γ	δ	$\alpha\beta$	$\alpha\beta\gamma$	$\alpha\beta\delta$	$(\alpha\beta)$; $(\gamma\delta)$; $[\alpha\beta]$	8	48
3321	3921		I	α	β	γ	$\alpha\beta\gamma$	$\alpha\beta\delta$	$(\alpha\beta)$	2	192
	4650		I	α	β	$\alpha\gamma$	$\beta\gamma$	$\alpha\beta\delta$	$(\alpha\beta)$	2	192
	5541	223	I	α	β	γ	$\alpha\beta$	$\alpha\beta\gamma\delta$	$(\alpha\beta)$	2	192
	5541	211	I	β	γ	$\alpha\beta$	$\alpha\gamma$	$\alpha\beta\delta$		1	384
3320	6720		I	α	β	$\alpha\beta$	$\alpha\gamma$	$\beta\gamma$	$(\alpha\beta)$; $[\alpha\beta]$	4	96
3311	6621		I	α	β	δ	$\alpha\beta$	$\alpha\beta\gamma$	$(\alpha\beta)$; $[\alpha\beta](\gamma\delta)$	4	96
3222	2931		I	δ	$\alpha\beta$	$\alpha\gamma$	$\alpha\delta$	$\beta\gamma$	$(\beta\gamma)$	2	192
	3660	2232	I	α	δ	$\alpha\beta$	$\beta\gamma$	$\alpha\gamma\delta$	$[\alpha](\beta\delta)$	2	192
	3660	3111	β	γ	δ	$\alpha\beta$	$\alpha\gamma$	$\alpha\delta$	$(\beta\gamma)$; $(\beta\delta)$; $[\alpha]$	12	32
	3822		I	α	β	δ	$\alpha\beta\gamma$	$\alpha\gamma\delta$	$(\beta\delta)$	2	192
	4551	2233	I	α	γ	δ	$\alpha\beta$	$\alpha\beta\gamma\delta$	$(\gamma\delta)$	2	192
	4551	2211	I	β	γ	$\alpha\gamma$	$\alpha\delta$	$\alpha\beta\delta$		1	384
	5442	2111	I	α	β	$\alpha\beta$	$\gamma\delta$	$\alpha\gamma\delta$	$(\gamma\delta)$; $[\alpha]$	4	96
	5442	1131	I	α	β	$\alpha\delta$	$\beta\gamma$	$\alpha\gamma\delta$	$[\alpha](\beta\delta)$	2	192
3221	4740	311	I	β	γ	$\alpha\beta$	$\alpha\gamma$	$\alpha\delta$	$(\beta\gamma)$	2	192
	4740	122	I	α	δ	$\alpha\beta$	$\alpha\gamma$	$\beta\gamma$	$(\beta\gamma)$	2	192
	5631	212	I	α	β	γ	$\alpha\beta$	$\alpha\gamma\delta$		1	384
	5631	121	I	α	β	δ	$\alpha\gamma$	$\alpha\beta\gamma$		1	384
3220	7620		I	α	β	γ	$\alpha\beta$	$\alpha\gamma$	$(\beta\gamma)$; $[\alpha]$	4	96
3211	6630		I	α	β	γ	$\alpha\beta$	$\alpha\delta$	$[\alpha](\gamma\delta)$	2	192
2222	4641	2333	I	α	β	γ	δ	$\alpha\beta\gamma\delta$	$(\alpha\beta)$; $(\alpha\gamma)$; $(\alpha\delta)$	24	16
	4641	1111	α	β	γ	δ	$\alpha\beta$	$\gamma\delta$	$(\alpha\beta)$; $(\alpha\gamma\beta\delta)$	8	48
	4641	1232	I	α	β	δ	$\alpha\gamma$	$\beta\gamma\delta$	$(\beta\delta)$	2	192
	4641	1212	I	α	γ	$\alpha\delta$	$\beta\gamma$	$\beta\delta$	$(\alpha\gamma)(\beta\delta)$	2	192
2221	4911		I	α	β	γ	δ	$\alpha\beta\gamma$	$(\alpha\beta)$; $(\alpha\gamma)$	6	64
	5640		I	β	γ	δ	$\alpha\beta$	$\alpha\gamma$	$(\beta\gamma)$	2	192
	6531		I	α	β	γ	$\alpha\beta$	$\gamma\delta$	$(\alpha\beta)$	2	192
2211	6720		I	α	β	γ	δ	$\alpha\beta$	$(\alpha\beta)$; $(\gamma\delta)$	4	96
											8008

Four Classes; Seven-fold Statements.

Letters	Distance	Basis	Type							Generators	Group Order	Conjugates
3333	0, 18, 03		I	$\alpha\beta$	$\alpha\gamma$	$\alpha\delta$	$\beta\gamma$	$\beta\delta$	$\gamma\delta$	$(\alpha\beta)$; $(\alpha\gamma)$; $(\alpha\delta)$	24	16
	2, 13, 42		α	β	γ	δ	$\alpha\beta$	$\alpha\gamma\delta$	$\beta\gamma\delta$	$(\alpha\beta)$; $(\gamma\delta)$	4	96
	3990		α	β	γ	$\alpha\delta$	$\beta\delta$	$\gamma\delta$	$\alpha\beta\gamma$	$(\alpha\beta)$; $(\alpha\gamma)$	6	64
	3, 10, 71		1	δ	$\alpha\beta$	$\alpha\gamma$	$\alpha\delta$	$\beta\gamma$	$\beta\gamma\delta$	$(\beta\gamma)$	2	192
	3, 12, 33		I	β	γ	δ	$\alpha\beta\gamma$	$\alpha\beta\delta$	$\alpha\gamma\delta$	$(\beta\gamma)$; $(\beta\delta)$	6	64
	4881	2222	α	β	γ	δ	$\alpha\beta$	$\gamma\delta$	$\alpha\beta\gamma\delta$	$(\alpha\beta)$; $(\alpha\gamma\beta\delta)$	8	48
	4881	2223	I	α	δ	$\alpha\beta$	$\beta\gamma$	$\alpha\gamma\delta$	$\beta\gamma\delta$		1	384
	4962	3232	I	β	δ	$\alpha\gamma$	$\alpha\delta$	$\beta\gamma$	$\alpha\beta\gamma\delta$	$(\alpha\gamma)(\beta\delta)$	2	192

Four Classes; Seven-fold Statements—(*Continued*).

Letters	Distance	Basis	Type							Generators	Group Order	Conjugates
	4962	1331	α	δ	$\alpha\beta$	$\alpha\gamma$	$\beta\gamma$	$\beta\delta$	$\gamma\delta$	$(\beta\gamma)$; $(\alpha\delta)$	4	96
	5772	2211	β	γ	δ	$\alpha\gamma$	$\alpha\delta$	$\beta\delta$	$\alpha\beta\gamma$		1	384
	5772	2222	I	β	γ	δ	$\alpha\delta$	$\alpha\beta\gamma$	$\alpha\beta\gamma\delta$	$(\beta\gamma)$	2	192
	5853		I	δ	$\alpha\gamma$	$\alpha\delta$	$\beta\gamma$	$\beta\delta$	$\alpha\beta\gamma$	$(\alpha\beta)$	2	192
	6663	2211	I	α	β	$\alpha\beta$	$\gamma\delta$	$\beta\gamma\delta$	$\alpha\gamma\delta$	$(\alpha\beta)$; $(\gamma\delta)$	4	96
	6663	1122	I	α	β	$\alpha\delta$	$\beta\gamma$	$\beta\gamma\delta$	$\alpha\gamma\delta$	$(\alpha\beta)(\gamma\delta)$	2	192
3332	3, 13, 32		I	α	$\alpha\beta$	$\alpha\gamma$	$\beta\gamma$	$\beta\delta$	$\gamma\delta$	$(\beta\gamma)$	2	192
	4, 10, 61	3323	I	α	β	γ	$\gamma\delta$	$\alpha\beta\gamma$	$\alpha\beta\delta$	$(\alpha\beta)$	2	192
	4, 10, 61	2212	α	β	γ	δ	$\alpha\gamma$	$\beta\gamma$	$\alpha\beta\delta$	$(\alpha\beta)$	2	192
	5871	1222	I	α	δ	$\alpha\beta$	$\alpha\gamma$	$\beta\gamma$	$\beta\gamma\delta$	$(\beta\gamma)$	2	192
	5871	1211	α	γ	δ	$\alpha\beta$	$\alpha\gamma$	$\beta\gamma$	$\beta\delta$	$(\alpha\gamma)$	2	192
	5871	2212	I	β	γ	$\alpha\gamma$	$\alpha\delta$	$\beta\delta$	$\alpha\beta\gamma$		1	384
	5871	2233	I	α	β	δ	$\alpha\gamma$	$\beta\gamma$	$\alpha\beta\gamma\delta$	$(\alpha\beta)$	2	192
	5952	2232	I	α	β	γ	$\alpha\beta$	$\alpha\gamma\delta$	$\beta\gamma\delta$	$(\alpha\beta)$	2	192
	5952	1232	I	α	γ	δ	$\alpha\beta$	$\alpha\beta\gamma$	$\beta\gamma\delta$		1	384
	5952	1121	α	β	γ	δ	$\alpha\beta$	$\gamma\delta$	$\alpha\beta\gamma$	$(\alpha\beta)$	2	192
	5952	2223	I	α	β	γ	δ	$\alpha\beta\gamma$	$\alpha\beta\gamma\delta$	$(\alpha\beta)$; $(\alpha\gamma)$	6	64
	6762	1121	I	α	δ	$\alpha\beta$	$\beta\gamma$	$\gamma\delta$	$\alpha\beta\gamma$		1	384
	6762	1123	I	α	β	$\alpha\gamma$	$\beta\gamma$	$\beta\delta$	$\alpha\gamma\delta$		1	384
	6762	2211	I	β	γ	$\alpha\gamma$	$\alpha\delta$	$\beta\gamma$	$\alpha\beta\delta$		1	384
	6762	1112	α	β	γ	$\alpha\beta$	$\alpha\gamma$	$\beta\delta$	$\gamma\delta$	$(\beta\gamma)$	2	192
	6762	2222	I	α	β	γ	$\alpha\delta$	$\beta\gamma$	$\alpha\beta\gamma\delta$	$(\beta\gamma)$	2	192
3331	4980		I	β	γ	$\alpha\beta$	$\alpha\gamma$	$\alpha\delta$	$\beta\gamma\delta$	$(\beta\gamma)$	2	192
	6960		α	β	γ	δ	$\alpha\beta$	$\alpha\gamma$	$\beta\gamma$	$(\alpha\beta)$; $(\alpha\gamma)$	6	64
	6, 10, 41		I	α	γ	$\alpha\beta$	$\alpha\gamma$	$\beta\gamma$	$\beta\delta$	$(\alpha\gamma)$	2	192
	7851	112	I	α	β	γ	$\alpha\beta$	$\gamma\delta$	$\alpha\beta\gamma$	$(\alpha\beta)$	2	192
	7851	111	I	α	β	δ	$\alpha\gamma$	$\beta\gamma$	$\alpha\beta\gamma$	$(\alpha\beta)$	2	192
	7851	122	I	α	β	γ	$\alpha\beta$	$\alpha\gamma$	$\beta\gamma\delta$	$(\beta\gamma)$	2	192
3330	9930		I	α	β	γ	$\alpha\beta$	$\alpha\gamma$	$\beta\gamma$	$(\alpha\beta)$; $(\alpha\gamma)$	6	64
3322	4, 13, 22		I	α	β	γ	δ	$\alpha\beta\gamma$	$\alpha\beta\delta$	$(\alpha\beta)$; $(\gamma\delta)$	4	96
	5970		I	α	β	γ	$\alpha\delta$	$\beta\delta$	$\alpha\beta\gamma$	$(\alpha\beta)$	2	192
	5, 10, 51		I	β	δ	$\alpha\beta$	$\alpha\gamma$	$\alpha\delta$	$\beta\gamma$		1	384
	6861	2233	I	α	β	γ	δ	$\alpha\beta$	$\alpha\beta\gamma\delta$	$(\alpha\beta)$; $(\gamma\delta)$	4	96
	6861	2112	I	β	γ	δ	$\alpha\beta$	$\alpha\gamma$	$\alpha\beta\delta$		1	384
	6861	1232	I	α	β	δ	$\alpha\beta$	$\alpha\gamma$	$\beta\gamma\delta$		1	384
	6861	1111	α	β	γ	δ	$\alpha\beta$	$\alpha\gamma$	$\beta\delta$	$(\alpha\beta)(\gamma\delta)$	2	192
	6942	1122	I	α	β	$\alpha\beta$	$\alpha\gamma$	$\beta\delta$	$\gamma\delta$	$(\alpha\beta)(\gamma\delta)$	2	192
	6942	1122	I	α	β	$\alpha\gamma$	$\alpha\delta$	$\beta\gamma$	$\beta\delta$	$(\alpha\beta)$; $(\gamma\delta)$	4	96
	7752	1211	I	α	β	γ	$\alpha\delta$	$\beta\gamma$	$\alpha\beta\delta$		1	384
	7752	1131	I	α	β	δ	$\alpha\beta$	$\gamma\delta$	$\alpha\beta\gamma$	$(\alpha\beta)$	2	192
3321	7950		I	α	β	δ	$\alpha\beta$	$\alpha\gamma$	$\beta\gamma$	$(\alpha\beta)$	2	192
	7, 10, 31		I	α	β	γ	δ	$\alpha\beta$	$\alpha\beta\gamma$	$(\alpha\beta)$	2	192
	8841		I	α	β	γ	$\alpha\beta$	$\alpha\delta$	$\beta\gamma$		1	384
3222	6960		I	β	γ	δ	$\alpha\beta$	$\alpha\gamma$	$\alpha\delta$	$(\beta\gamma)$; $(\beta\delta)$	6	64
	6, 10, 41		I	α	β	γ	δ	$\alpha\delta$	$\alpha\beta\gamma$	$(\beta\gamma)$	2	192
	7851		I	α	β	γ	$\alpha\beta$	$\alpha\delta$	$\gamma\delta$		1	384
3221	8940		I	α	β	γ	δ	$\alpha\beta$	$\alpha\gamma$	$(\beta\gamma)$	2	192
2222	8841		I	α	β	γ	δ	$\alpha\beta$	$\gamma\delta$	$(\alpha\beta)$; $(\alpha\gamma\beta\delta)$	8	48
												11,440

Four Classes; Eight-fold Statements.

Letters	Distance	Basis	Type							
4444	0, 24, 04		I	$\alpha\beta$	$\alpha\gamma$	$\alpha\delta$	$\beta\gamma$	$\beta\delta$	$\gamma\delta$	$\alpha\beta\gamma\delta$
	4, 12, 12, 0		I	γ	$\alpha\beta$	$\alpha\delta$	$\beta\delta$	$\alpha\beta\gamma$	$\alpha\gamma\delta$	$\beta\gamma\delta$
	4, 14, 82		I	$\alpha\gamma$	$\alpha\delta$	$\beta\gamma$	$\beta\delta$	$\gamma\delta$	$\alpha\beta\gamma$	$\alpha\beta\delta$
	6, 10, 10, 2		I	γ	$\alpha\gamma$	$\alpha\delta$	$\beta\delta$	$\alpha\beta\gamma$	$\alpha\beta\delta$	$\beta\gamma\delta$
	6, 12, 64		I	α	β	γ	$\alpha\beta\delta$	$\alpha\gamma\delta$	$\beta\gamma\delta$	$\alpha\beta\gamma\delta$
	8884	2662	I	β	γ	$\alpha\delta$	$\beta\gamma$	$\alpha\beta\delta$	$\alpha\gamma\delta$	$\alpha\beta\gamma\delta$
	8884	4444	I	α	β	$\alpha\delta$	$\beta\gamma$	$\alpha\gamma\delta$	$\beta\gamma\delta$	$\alpha\beta\gamma\delta$
4443	3, 18, 43		I	γ	$\alpha\beta$	$\alpha\gamma$	$\alpha\delta$	$\beta\gamma$	$\beta\delta$	$\alpha\beta\gamma\delta$
	5, 12, 10, 1		I	α	$\alpha\gamma$	$\beta\gamma$	$\beta\delta$	$\gamma\delta$	$\alpha\beta\gamma$	$\alpha\beta\delta$
	6, 11, 92	4443	β	γ	$\alpha\beta$	$\alpha\gamma$	$\alpha\delta$	$\beta\delta$	$\gamma\delta$	$\alpha\beta\gamma$
	6, 11, 92	3333	I	γ	δ	$\alpha\beta$	$\alpha\gamma$	$\beta\gamma$	$\alpha\beta\delta$	$\alpha\beta\gamma\delta$
	7, 10, 83	3331	I	α	β	$\alpha\beta$	$\gamma\delta$	$\alpha\beta\gamma$	$\alpha\gamma\delta$	$\beta\gamma\delta$
	7, 10, 83	3412	I	β	δ	$\alpha\gamma$	$\alpha\delta$	$\beta\gamma$	$\alpha\beta$	$\alpha\beta\gamma\delta$
4442	6, 12, 10, 0		I	β	γ	$\alpha\beta$	$\alpha\gamma$	$\alpha\delta$	$\alpha\beta\gamma$	$\beta\gamma\delta$
	6, 14, 62		I	γ	$\alpha\beta$	$\alpha\gamma$	$\alpha\delta$	$\beta\gamma$	$\beta\delta$	$\alpha\beta\gamma$
	8, 10, 82	3421	I	α	γ	$\alpha\gamma$	$\beta\gamma$	$\beta\delta$	$\alpha\beta\gamma$	$\alpha\beta\delta$
	8, 10, 82	4462	I	α	β	γ	$\alpha\beta$	$\gamma\delta$	$\alpha\beta\gamma$	$\alpha\beta\gamma\delta$
	8, 10, 82	4413	I	α	β	δ	$\alpha\gamma$	$\beta\gamma$	$\alpha\beta\gamma$	$\alpha\beta\gamma\delta$
4441	9, 12, 61		I	β	γ	$\alpha\beta$	$\alpha\gamma$	$\alpha\delta$	$\beta\gamma$	$\alpha\beta\gamma$
4440	12, 12, 40		I	α	β	γ	$\alpha\beta$	$\alpha\gamma$	$\beta\gamma$	$\alpha\beta\gamma$
4433	5, 14, 72		I	β	γ	$\alpha\beta$	$\alpha\gamma$	$\alpha\delta$	$\beta\delta$	$\alpha\beta\gamma\delta$
	6, 11, 10, 1	3322	I	β	γ	$\alpha\beta$	$\alpha\gamma$	$\alpha\delta$	$\alpha\beta\delta$	$\beta\gamma\delta$
	6, 11, 10, 1	6432	I	β	γ	$\alpha\beta$	$\alpha\delta$	$\beta\delta$	$\alpha\beta\gamma$	$\alpha\gamma\delta$
	6, 13, 63	3311	I	γ	$\alpha\beta$	$\alpha\gamma$	$\alpha\delta$	$\beta\gamma$	$\beta\delta$	$\alpha\beta\delta$
	6, 13, 63	3333	I	α	β	γ	$\alpha\beta$	$\alpha\beta\delta$	$\alpha\gamma\delta$	$\beta\gamma\delta$
	6, 13, 63	3133	I	β	γ	δ	$\alpha\beta$	$\alpha\beta\gamma$	$\alpha\beta\delta$	$\alpha\gamma\delta$
	6, 13, 63	1322	I	α	$\alpha\beta$	$\alpha\gamma$	$\alpha\delta$	$\beta\gamma$	$\beta\delta$	$\beta\gamma\delta$
	7, 10, 92	1422	I	α	δ	$\alpha\beta$	$\beta\gamma$	$\gamma\delta$	$\alpha\beta\gamma$	$\alpha\beta\delta$
	7, 10, 92	3411	I	α	γ	$\alpha\delta$	$\beta\gamma$	$\beta\delta$	$\alpha\beta\gamma$	$\alpha\beta\delta$
	7, 10, 92	4322	I	α	β	$\alpha\beta$	$\alpha\delta$	$\gamma\delta$	$\alpha\beta\gamma$	$\beta\gamma\delta$
	8983	4411	I	γ	δ	$\alpha\beta$	$\alpha\delta$	$\beta\gamma$	$\alpha\beta\gamma$	$\alpha\beta\delta$
	8983	4422	I	α	β	$\alpha\beta$	$\alpha\delta$	$\beta\gamma$	$\alpha\gamma\delta$	$\beta\gamma\delta$
	8983	6212	I	β	γ	$\alpha\beta$	$\alpha\gamma$	$\beta\delta$	$\alpha\beta\delta$	$\alpha\gamma\delta$
4432	7, 12, 81	3412	I	β	δ	$\alpha\beta$	$\alpha\gamma$	$\alpha\delta$	$\beta\gamma$	$\alpha\beta\delta$
	7, 12, 81	4433	I	α	β	γ	$\alpha\beta$	$\alpha\delta$	$\alpha\beta\gamma$	$\beta\gamma\delta$
	8, 11, 72	3211	I	β	δ	$\alpha\beta$	$\alpha\gamma$	$\alpha\delta$	$\beta\gamma$	$\alpha\beta\gamma$
	8, 11, 72	1322	I	α	β	δ	$\alpha\beta$	$\alpha\gamma$	$\alpha\beta\gamma$	$\beta\gamma\delta$
	8, 11, 72	4412	I	α	β	$\alpha\gamma$	$\alpha\delta$	$\beta\gamma$	$\beta\delta$	$\alpha\beta\gamma$
	8, 11, 72	1423	I	α	β	$\alpha\beta$	$\alpha\gamma$	$\alpha\delta$	$\beta\gamma$	$\beta\gamma\delta$
4431	10, 11, 61		I	α	β	δ	$\alpha\beta$	$\alpha\gamma$	$\beta\gamma$	$\alpha\beta\gamma$
4422	8, 12, 80		I	α	β	δ	$\alpha\beta$	$\alpha\gamma$	$\beta\gamma$	$\alpha\beta\delta$
	8, 14, 42		I	α	β	$\alpha\beta$	$\alpha\gamma$	$\alpha\delta$	$\beta\gamma$	$\beta\delta$
	10, 10, 62		I	α	β	δ	$\alpha\beta$	$\alpha\gamma$	$\beta\delta$	$\alpha\beta\gamma$
4333	4, 18, 33		I	α	$\alpha\beta$	$\alpha\gamma$	$\alpha\delta$	$\beta\gamma$	$\beta\delta$	$\gamma\delta$
	6, 12, 91	5222	I	β	γ	δ	$\alpha\beta$	$\alpha\gamma$	$\alpha\delta$	$\alpha\beta\gamma\delta$
	6, 12, 91	3223	I	α	β	δ	$\alpha\gamma$	$\beta\gamma$	$\alpha\beta\delta$	$\alpha\gamma\delta$
	7, 11, 82	3211	I	γ	δ	$\alpha\beta$	$\alpha\gamma$	$\alpha\delta$	$\beta\delta$	$\alpha\beta\gamma$
	7, 11, 82	6311	I	β	γ	δ	$\alpha\gamma$	$\alpha\delta$	$\alpha\beta\gamma$	$\alpha\beta\delta$
	7, 11, 82	4322	I	α	β	δ	$\alpha\gamma$	$\alpha\delta$	$\alpha\beta\gamma$	$\beta\gamma\delta$
	7, 11, 82	1222	I	α	δ	$\alpha\beta$	$\alpha\gamma$	$\alpha\delta$	$\beta\gamma$	$\beta\gamma\delta$

Four Classes; Eight-fold Statements—*Continued*).

Letters	Distance	Basis	Type							
	8, 10, 73	4211	I	α	δ	$\alpha\gamma$	$\alpha\delta$	$\beta\gamma$	$\beta\delta$	$\alpha\beta\gamma$
	8, 10, 73	2212	I	α	γ	$\alpha\beta$	$\alpha\delta$	$\beta\gamma$	$\gamma\delta$	$\alpha\beta\delta$
4332	7, 14, 52		I	α	γ	$\alpha\beta$	$\alpha\gamma$	$\alpha\delta$	$\beta\gamma$	$\beta\delta$
	8, 11, 81	3111	I	β	γ	δ	$\alpha\beta$	$\alpha\gamma$	$\alpha\delta$	$\alpha\beta\gamma$
	8, 11, 81	1223	I	α	β	γ	$\alpha\beta$	$\alpha\gamma$	$\alpha\delta$	$\beta\gamma\delta$
	8, 11, 81	4122	I	α	β	δ	$\alpha\beta$	$\alpha\gamma$	$\beta\gamma$	$\alpha\gamma\delta$
	9, 10, 72	4211	I	α	β	γ	$\alpha\gamma$	$\alpha\delta$	$\beta\gamma$	$\alpha\beta\delta$
	9, 10, 72	4111	I	α	β	δ	$\alpha\gamma$	$\alpha\delta$	$\beta\gamma$	$\alpha\beta\gamma$
	9, 10, 72	4113	I	α	β	γ	$\alpha\beta$	$\alpha\gamma$	$\beta\delta$	$\alpha\gamma\delta$
4331	10, 12, 51		I	α	β	γ	δ	$\alpha\gamma$	$\beta\gamma$	$\alpha\beta\gamma$
4322	9, 12, 61		I	α	β	δ	$\alpha\beta$	$\alpha\gamma$	$\alpha\delta$	$\beta\gamma$
4222	10, 12, 60		I	α	β	γ	δ	$\alpha\beta$	$\alpha\gamma$	$\alpha\delta$
3333	6, 12, 10, 0		I	β	γ	δ	$\alpha\beta$	$\alpha\gamma$	$\alpha\delta$	$\beta\gamma\delta$
	6, 14, 62	3311	I	γ	δ	$\alpha\beta$	$\alpha\gamma$	$\alpha\delta$	$\beta\gamma$	$\beta\delta$
	6, 14, 62	2233	I	α	β	γ	δ	$\alpha\gamma$	$\alpha\beta\delta$	$\beta\gamma\delta$
	8, 10, 82	2211	I	β	γ	δ	$\alpha\gamma$	$\alpha\delta$	$\beta\gamma$	$\alpha\beta\delta$
	8, 10, 82	2222	I	α	β	γ	δ	$\alpha\beta$	$\gamma\delta$	$\alpha\beta\gamma\delta$
	8, 10, 82	1111	α	β	γ	δ	$\alpha\beta$	$\alpha\gamma$	$\beta\delta$	$\gamma\delta$
3332	8, 12, 71	3111	I	β	γ	δ	$\alpha\beta$	$\alpha\gamma$	$\alpha\delta$	$\beta\gamma$
	8, 12, 71	2122	I	α	β	γ	δ	$\alpha\gamma$	$\beta\gamma$	$\alpha\beta\delta$
	9, 11, 62	2111	I	α	β	γ	δ	$\alpha\gamma$	$\beta\delta$	$\alpha\beta\delta$
	9, 11, 62	1112	I	α	β	δ	$\alpha\beta$	$\alpha\gamma$	$\beta\delta$	$\gamma\delta$
3331	10, 12, 60		I	α	β	γ	δ	$\alpha\beta$	$\alpha\gamma$	$\beta\gamma$
3322	10, 11, 61		I	α	β	γ	δ	$\alpha\beta$	$\alpha\gamma$	$\gamma\delta$

Four Classes; Eight-fold Statements.

Distance	Generators	Group Order	Conjugates
0, 24, 04	(all); $[\alpha\beta]$; $[\alpha\gamma]$; $[\alpha\delta]$	192	2
4, 12, 12, 0	$(\alpha\beta)$; $(\alpha\delta)$; [all]	48	8
4, 14, 82	$(\alpha\beta)$; $(\gamma\delta)$; $[\gamma\delta]$	8	48
6, 10, 10, 2	$[\alpha\delta](\beta\gamma)$; $[\beta\gamma\delta]$	4	96
6, 12, 64	$(\alpha\beta)$; $(\alpha\gamma)$; $[\alpha\beta\gamma\delta]$	12	32
8884	$(\beta\gamma)$; $(\alpha\delta)$; [all]	32	12
8884	$[\alpha](\alpha\beta\gamma\delta)$; $[\alpha](\beta\delta)$; $[\alpha\beta\gamma\delta]$	16	24
3, 18, 43	$[\alpha\gamma](\alpha\gamma)$; $[\beta\gamma](\beta\gamma)$;	6	64
5, 12, 10, 1	$[\beta\gamma](\beta\gamma)$	2	192
6, 11, 92	$(\beta\gamma)$	2	192
6, 11, 92	$(\alpha\beta)$	2	192
7, 10, 83	$(\alpha\beta)$	2	192
7, 10, 83		1	384
6, 12, 10, 0	$(\beta\gamma)$; $[\alpha\beta](\alpha\beta)$; $[\alpha\beta\gamma]$	12	32
6, 14, 62	$(\alpha\beta)$; $[\alpha\beta]$	4	96
8, 10, 82	$[\beta\gamma](\beta\gamma)$; $[\alpha]$	4	96
8, 10, 82	$(\alpha\beta)$; $[\alpha\beta]$	4	96
8, 10, 82	$(\alpha\beta)$; $[\alpha\beta\gamma]$	4	96
9, 12, 61	$(\beta\gamma)$; $[\alpha\beta](\alpha\beta)$	6	64
12, 12, 40	$(\alpha\beta)$; $(\alpha\gamma)$; [all]	48	8
5, 14, 72		1	384
6, 11, 10, 1	$[\alpha\beta](\alpha\beta)(\gamma\delta)$	2	192
6, 11, 10, 1	$[\alpha\beta](\alpha\beta)$	2	192
6, 13, 63	$(\alpha\beta)$; $[\alpha\beta](\gamma\delta)$	4	96

Four Classes; Eight-fold Statements—(*Continued*).

Distance	Generators	Group Order	Conjugates
6, 13, 63	$(\alpha\beta)$; $[\alpha\beta](\gamma\delta)$	4	96
6, 13, 63	$(\gamma\delta)$; $[\alpha\beta](\alpha\beta)$	4	96
6, 13, 63	$(\gamma\delta)$; $[\alpha\beta](\alpha\beta)$	4	96
7, 10, 92		1	384
7, 10, 92	$[\alpha](\gamma\delta)$	2	192
7, 10, 92	$[\beta](\gamma\delta)$	2	192
8983	$(\alpha\beta)(\gamma\delta)$; $[\alpha\beta](\alpha\beta)$	4	96
8983	$(\alpha\beta)(\gamma\delta)$; $[\alpha\beta](\gamma\delta)$	4	96
8983		1	384
7, 12, 81	$[\alpha\beta](\alpha\beta)$	2	192
7, 12, 81	$[\alpha\beta](\alpha\beta)$	2	192
8, 11, 72		1	384
8, 11, 72		1	384
8, 11, 72	$(\alpha\beta)$	2	192
8, 11, 72	$[\alpha\beta](\alpha\beta)$	2	192
10, 11, 61	$(\alpha\beta)$	2	192
8, 12, 80	$(\alpha\beta)$; $[\alpha](\gamma\delta)$; $[\alpha\beta]$	8	48
8, 14, 42	$(\alpha\beta)$; $(\gamma\delta)$; $[\alpha\beta]$	8	48
10, 10, 62	$(\alpha\beta)(\gamma\delta)$; $[\beta]$	4	96
4, 18, 33	$(\beta\gamma)$; $(\gamma\delta)$	6	64
6, 12, 91	$(\beta\gamma)$; $(\gamma\delta)$	6	64
6, 12, 91		1	384
7, 11, 82		1	384
7, 11, 82	$(\gamma\delta)$	2	192
7, 11, 82		1	384
7, 11, 82	$(\beta\gamma)$	2	192
8, 10, 73		1	384
8, 10, 73	$(\alpha\delta)$	2	192
7, 14, 52		1	384
8, 11, 81	$(\beta\gamma)$	2	192
8, 11, 81	$(\beta\gamma)$	2	192
8, 11, 81		1	384
9, 10, 72		1	384
9, 10, 72	$[\alpha](\beta\gamma)$	2	192
9, 10, 72	$[\alpha](\beta\gamma)$	2	192
10, 12, 51	$(\alpha\beta)$	2	192
9, 12, 61		1	384
10, 12, 60	$(\beta\delta)$; $(\gamma\delta)$; $[\alpha]$	12	32
6, 12, 10, 0	$(\beta\gamma)$; $(\gamma\delta)$	6	64
6, 14, 62	$(\alpha\beta)$; $(\gamma\delta)$	4	96
6, 14, 62	$(\alpha\gamma)$; $(\beta\delta)$	4	96
8, 10, 82		1	384
8, 10, 82	$(\alpha\beta)$; $(\alpha\gamma\beta\delta)$	8	48
8, 10, 82	$(\alpha\delta)$; $(\alpha\gamma\delta\beta)$	8	48
8, 12, 71	$(\beta\gamma)$	2	192
8, 12, 71	$(\alpha\beta)$	2	192
9, 11, 62	$(\beta\delta)$	2	192
9, 11, 62	$(\alpha\delta)$	2	192
10, 12, 60	$(\alpha\beta)$; $(\alpha\gamma)$	6	64
10, 11, 61	$(\alpha\gamma)(\beta\delta)$	2	192
			12,870

A Memoir on the Invariants of Biternary Quantics.*

By O. E. Glenn.

An object of this paper is to apply transvection † as the fundamental process in methods which may be employed, in the case of any connex $f = a_x^m \alpha_u^n$, to find the complete system of invariant formations. The results possess, for this object, a generality not before attained. Use is made of analogies with methods current in the binary field, for instance a new result reached is a theory of constant symbolical moduli for biternary forms. The applications treated relate to the system of $a_x^2 \alpha_u$. The system of $a_x \alpha_u$ is well known.‡ By a process of investigation from one degree to the next higher we obtain the complete system of 32 concomitants of $a_x^2 \alpha_u$ of the first three degrees. Maisano gave,§ quite analogously, the complete system of the first five degrees of the ternary quartic. Another result reached is a system of 90 concomitants of $a_x^2 \alpha_u$ which is relatively finite and complete with regard to a set of symbolical moduli. This result is comparable, in principle, with E. Noether's system ‖ for the ternary quartic, which is also relatively finite and complete with respect to a set of moduli. We also obtain a formulary and theory which constitute a noteworthy generalization of the reducing series known in the binary theory as Gordan's series.¶

1. *Introduction.* A biternary transvectant T of two forms,

$$\phi = a_{1x} a_{2x} \cdots a_{rx} \alpha_{1u} \alpha_{2u} \cdots \alpha_{su},$$
$$\psi = b_{1x} b_{2x} \cdots b_{\rho x} \beta_{1u} \beta_{2u} \cdots \beta_{\sigma u},$$

is characterized by four indices i, j, k, l, and is the result of an analytical operation consisting of multiple polarization. The definitions formulated

* Presented to the American Mathematical Society, May and September, 1926.

† Gordan, *Journal für Mathematik*, Vol. 69 (1868), p. 323; Cayley, *Mathematical Papers*, Vol. 7, p. 348; Isserlis, *American Journal of Mathematics*, Vol. 45 (1923), p. 54.

‡ Clebsch and Gordan, *Mathematische Annalen*, Vol. 1 (1869), p. 359; Clebsch, *Vorlesungen über Geometrie*, 1 Aufl., p. 942.

§ *Giornali di Battaglini*, Vol. 19 (1881), p. 198.

‖ *Journal für Mathematik*, Vol. 134 (1908), p. 23.

¶ *Mathematische Annalen*, Vol. 5 (1872), p. 95; Grace and Young, *Algebra of Invariants*, 1st Ed. (1903), p. 56.

from this initial point of view are to be found in a previous paper by the present writer.* An equivalent definition is the following: Given two forms,

$$\phi = a_x{}^r \alpha_u{}^s, \quad \psi = b_x{}^\rho \beta_u{}^\sigma,$$

then, by definition,†

$$T = (\phi, \psi)\,[ijkl] = a_\beta{}^i b_a{}^k\,(abu)^j (\alpha\beta x)^l\; a_x{}^{r-i-j} b_x{}^{\rho-j-k}\; \alpha_u{}^{s-k-l} \beta_u{}^{\sigma-i-l},$$

and we have, necessarily,

$$(1) \qquad i+j \leqq r, \quad k+l \leqq s, \quad i+l \leqq \sigma, \quad j+k \leqq \rho.$$

The integer $g = r+s+\rho+\sigma$, which is the sum of the order and the class of the product $\phi\psi$, is called the grade of T.

Accordingly a transvectant T is a polynomial in expressions formed by means of seven types of factors, viz., a_x, α_u, a_a, (abc), $(\alpha\beta\gamma)$, (abu), $(\alpha\beta x)$, types which can be produced within a single form (connex)

$$C = A a_{1x} \cdots a_{rx} \alpha_{1u} \cdots \alpha_{su},$$

by use of three operations of convolution, thus,

$$C_1 = A(a_1 a_2 u) a_{3x} \cdots a_{rx} \alpha_{1u} \cdots \alpha_{su},$$
$$C_2 = A(\alpha_1 \alpha_2 x) a_{1x} \cdots a_{rx} \alpha_{3u} \cdots \alpha_{su},$$
$$C_3 = A a_{1a_1} a_{2x} \cdots a_{rx} \alpha_{2u} \cdots \alpha_{su}.$$

For example the factor $(a_1 a_2 a_3)$ is obtained from C_1 followed by the type C_3, after writing $(a_1 a_2 u) = (a_1 a_2)_u$. The result of applying, to C, ζ convolutions of the first type (C_1), η of the second and θ of the third will be designated in this paper by,

$$(2) \qquad \zeta\eta\theta/C.$$

2. *The definitive process of construction.* The theory of ternary transvection was treated, in its relation to simultaneous systems, in two previous papers by the present author. The system of the pair ‡ $p_x{}^2$, $a_x\alpha_u$ and that of the pair § $a_x\alpha_u$, $p_x\theta_u$ were found. That transvection is a definitive process for the generation of all concomitants of a single connex is a result of the following:

---

* *Transactions of the American Mathematical Society*, Vol. 17 (1916), p. 405.

† In previous papers this formula was written $(\phi, \psi)^{ij}_{kl}$. Cf. This JOURNAL, Vol. 48, p. 45.

‡ *Transactions of the American Mathematical Society, loc. cit.*, p. 412.

§ *American Journal of Mathematics*, Vol. 48 (1926), pp. 45 to 56.

THEOREM. *Every monomial mixed concomitant t of a connex $\phi = a_x^m \alpha_u^n$, (including invariants and covariants as special cases), is a term of a transvectant $T = (P, Q)[ijkl]$, where $P = a_\alpha^p a_x^{m-p} \alpha_u^{n-p}$ and Q is a monomial concomitant of a degree (in the coefficients of ϕ) one less than the degree of t.*

The most general form of t, omitting numerical factors, may be written in three parts,

$$t = a_\alpha^p a_x^\mu \alpha_u^\nu \times \Pi(abc)\Pi a_\beta \Pi b_\alpha \; \Pi(\alpha\beta\gamma)\Pi(abu)\Pi(\alpha\beta x) \times F(b, c, \cdots, \beta, \gamma, \cdots),$$

in which a Π function represents the product of all factors in t of the indicated type and the symbols are equivalent according to the rule $a_x^m \alpha_u^n = b_x^m \beta_u^n = \cdots$. The first part contains those factors of t that involve a or α with no other symbols, the second part contains all symbolic factors that involve either a or α with any of $b, c, \cdots, \beta, \gamma, \cdots$, and F includes all factors that do not contain a or α.

In the second part of t we are to replace a by u and α by x but for practical reasons it is desirable first to replace a by v and α by y where v is cogredient to u and y is cogredient to x, obtaining a preliminary form,

$$t_{(1)} = a_\alpha^p a_x^\mu \alpha_u^\nu \times \Pi(bcv)\Pi\beta_v \Pi b_y \; \Pi(\beta\gamma y)\Pi(buv)\Pi(\beta x y) \times F(b, \cdots; \beta, \cdots).$$

As is well known,* $(buv) = b_x$, $(\beta x y) = \beta_u$. Suppose that the number of factors (bcv), β_v in $t_{(1)}$ is i, that there are j factors (buv), k of the types $(\beta\gamma y)$, b_y and l of the type $(\beta x y)$. Then t is a term of

$$T_1 = (P, \Pi(bcu)\Pi_1 \beta_u \Pi_1 b_x \; \Pi(\beta\gamma x)\Pi_2 b_x \Pi_2 \beta_u F(b, c, \cdots; \beta, \gamma, \cdots))[ijkl].$$

Note that the first product of factors β_u, i. e. $\Pi_1\beta_u$, was derived from $\Pi\beta_v$ while the second product $\Pi_2\beta_u$ was derived from $\Pi(\beta x y)$: Likewise in the case of $\Pi_1 b_x$, $\Pi_2 b_x$. The theorem is proved.

We refer to the process which produces $t_{(1)}$ from t as contraction with regard to $a_x^m \alpha_u^n$ or with reference to the pair of symbols (a, α).

In the analogous manner we may contract any monomial with regard to any set of symbolic pairs, $\{(a, \alpha), (b, \beta), \cdots\}$, which are involved in it.

The sum of the numerical coefficients in any T is unity and it follows from theorems in my former paper (*Transactions of the American Mathematical Society*), quoted above, that, if t is a term of T, the difference $t - T$ equals a linear combination of terms each of which is a term of a transvectant

* Grace and Young, *loc. cit.*, p. 258.

of grade $< g$ whose two forms are forms derived by convolution from ϕ and ψ respectively. If t' is a term of $t - T$ and if t' is a term of T', we have,

$$t = T + \Sigma ct' = T + \Sigma cT' + \Sigma dt'', \tag{3}$$

t'' being a term of a transvectant of grade less than that of T'. Since the grade can be reduced only a finite number of times the arbitrary monomial t is ultimately expressed as a linear combination of transvectants. Hence,

THEOREM. *By the operation of transvection we can generate all invariant formations in the system of a connex.*

3. *Inductive procedure relating to the degrees.* One method of procedure, if the problem is to find a fundamental system of $\phi = a_x{}^m \alpha_u{}^n$, is to construct an irreducible set, containing the maximum number, of degrees $\leqq d - 1$ and proceed by induction to the construction of a complete set of irreducible forms of degree d. This induction starts with the following complete set of degree one, $(n \leqq m)$:

$$\phi_p = a_a{}^p a_x{}^{m-p} \alpha_u{}^{n-p}, \quad (p = 0, \cdots, n). \tag{4}$$

Let the maximal set of degrees $\leqq d - 1$ be,

$$\gamma_1, \gamma_2, \cdots, \gamma_s.$$

Any form of degree d is a linear combination of monomials t and hence of transvectants,

$$T_1 = (\phi_p, G)[ijkl],$$

and G, being of degree $d - 1$, is a polynomial in $\gamma_1, \cdots, \gamma_s$.

It has been shown that if $[ijkl] = [m + q, n + r, o + s, p + t]$ and $\phi = AB$, $\psi = CD$, all factors being equal to non-symbolical quantics, while $(A, C)[mnop]$, $(B, D)[qrst]$ exist and are not zero, $T = (\phi, \psi)[ijkl]$ is reducible (*Transactions of the American Mathematical Society, loc. cit.*, p. 409).

Now the transvectant $(\phi_p, G)[ijkl]$ is a linear combination of the transvectants, of the same matrix, of ϕ_p and the various terms of G. Hence we have only to find the transvectants of the formula,

$$T_1 = (\phi_p, \gamma_1{}^{\sigma_1} \gamma_2{}^{\sigma_2} \cdots \gamma_s{}^{\sigma_s})[ijkl],$$

which are not reducible by the process of separation just described. It is known that the number of these in any case is finite. Hence when these are found the induction will be complete.

In § 7 of this paper this method is applied to the special orders (2, 1).

4. *Biternary reducing series.* Previous writers have considered forms of generalization * of the finite series of Gordan which expresses a doubly binary quantic as a power series in the determinant (xy). A modification of this series reduces it to a linear relation between transvectants † of the type $(f,(g,h)^r)^s$ and this form of the series may be effectively generalized for biternary transvectants T. The method leads to a very general reducing series by means of which reducible biternary transvectants of forms of numerical orders (m,n) may be reduced by substitutions in a formula. The method of proof of the series is also of an inductive nature such that it applies to the direct construction of the special forms of the reducing series which are required in order to reduce special transvectants. We therefore also arrive at an important practical process of performing reductions. We treat two cases of some complexity in full.

Let

$$f = a_x^2\alpha_u = b_x^2\beta_u, \quad g = a_\alpha a_x = b_\beta b_x.$$

Then,

$$(f,f)\,[0110] = (bcu)\,c_\beta b_x \gamma_u.$$

Consider the form,

$$T = (g,\,(f,f)\,[0110])\,[1000].$$

It has the term $t = a_\alpha c_\beta (abc)\gamma_u b_x$. But this is also a term of ‡

$$S = -\,(f,\,(g,f)\,[0100])\,[1010].$$

Successive use of the theorem on the difference between a transvectant and one of its terms will give, therefore, an equality between two finite series whose terms are transvectants, and each series is equal to t. Thus, (cf. (3)),

$$pT + p_1T_1 + \cdots + p_sT_s = qS + q_1S_1 + \cdots + q_tS_t.$$

The order of arrangement of the quantics g, f, f in any T_i is the same as in T and the order in any S_i is the same as in S. To derive this series we have

$$(5) \qquad T = (a_\alpha a_x,\, c_\beta b_x \gamma_u (bcu))\,[1000] = \tfrac{1}{2}\{t + a_\alpha c_\beta a_\gamma b_x (bcu)\}.$$
$$T - t = \tfrac{1}{2} a_\alpha c_\beta b_x \{(bc)_u \gamma_a - (bc)_a \gamma_u\} = \tfrac{1}{2} a_\alpha c_\beta b_x (\{(bc)\gamma\} ua) = \tfrac{1}{2}s.$$

---

* Study, *Methoden zur Theorie der ternären Formen* (1889), p. 54; Capelli, *Giornali di Battaglini*, Vol. 18 (1880), p. 17; E. Noether, *Journal für Mathematik*, Vol. 134 (1908), p. 29.

† Grace and Young, *loc. cit.*, p. 56.

‡ Contract t with regard to $b_x^2\beta_u$.

Now s is a term of

$$T_1 = (a_\alpha a_x,\ c_\beta b_x(\{bc\}\gamma x))\,[0100] = \tfrac{1}{2}\{s + a_\alpha c_\beta(abu)\,(\{bc\}\gamma x)\},$$

and

$$T_1 - s = \tfrac{1}{2}(a_\alpha a_x,\ (b\{(bc)\gamma\}u)c_\beta)\,[1000]u_x + \tfrac{1}{2}(a_\alpha a_x,\ (\{(bc)\gamma\}bu)c_\beta)\,[0000]$$
$$= \tfrac{1}{2}T_2u_x + \tfrac{1}{2}T_3.$$

We obtain, likewise, from S,

$$S = (b_x{}^2\beta_u,\ a_\alpha c_x\gamma_u(cau))\,[1010] = \tfrac{1}{2}\{t + a_\alpha c_\beta b_\gamma b_x(cau)\}.$$
$$S - t = \tfrac{1}{2}a_\alpha c_\beta b_x\{(ca)_u\gamma_b - (ca)_b\gamma_u\} = \tfrac{1}{2}a_\alpha c_\beta b_x(\{(ca)\gamma\}ub) = \tfrac{1}{2}\tau.$$

Also τ is a term of

$$S_1 = (b_x{}^2\beta_u, a_\alpha c_x(\{ca\}\gamma x))\,[0110] = \tfrac{1}{2}\{\tau + a_\alpha b_x(\{ca\}\gamma\beta)\,(bcu)\},$$

as is shown by contracting τ with regard to $b_x{}^2\beta_u$. Hence,

$$S_1 - \tau = \tfrac{1}{2}a_\alpha b_x\{(\{ca\}\gamma)_\beta(bcu) - c_\beta(b\{(ca)\gamma\}u)\}$$
$$= \tfrac{1}{2}a_\alpha b_x\{(\{(ca)\gamma\}bc)u_\beta + (\{(ca)\gamma\}cu)b_\beta\}$$
$$= \tfrac{1}{2}(b_x{}^2\beta_u, a_\alpha(c\{(ca)\gamma\}u))\,[1000] + \tfrac{1}{2}(b_\beta b_x,\ a_\alpha(\{(ca)\gamma\}cu))\,[0000]$$
$$= \tfrac{1}{2}S_2 + \tfrac{1}{2}S_3.$$

Equating the value of t given by the formulas relating to T to the value of t obtained from S we get the reducing sequence * for T, S, viz.,

$$(6) \qquad T - \tfrac{1}{2}T_1 + \tfrac{1}{4}T_2u_x + \tfrac{1}{4}T_3 = S - \tfrac{1}{2}S_1 + \tfrac{1}{4}S_2 + \tfrac{1}{4}S_3.$$

The analogy with Gordan's series may be emphasized by writing, according to the general theory, the second form in each transvectant T_i as a result of applying convolutions to $(f,f)\,[0110]$ and the second form in each S_i as a convolution from $(g,f)\,[0100]$, under the notation (2). It will be understood that the ζ, in $\zeta\eta\theta/C$, produces only factors of the type (abu) and the η, factors of the type $(\alpha\beta x)$ only. Our series may then be written,

$$(g,\ (f,f)\,[0110])\,[1000] - \tfrac{1}{2}(g, 010/(f,f)\,[0110])\,[0100]$$
$$+ \tfrac{1}{4}(g, 110/(f,f)\,[0110])\,[1000]u_x$$
$$+ \tfrac{1}{4}(g, 110/(f,f)\,[0110])\,[0000]$$
$$= -(f,\ (g,f)\,[0100])\,[1010] + \tfrac{1}{2}(f, 010/(g,f)\,[0100])\,[0110]$$
$$- \tfrac{1}{4}(f, 110/(g,f)\,[0100])\,[1000]$$
$$- \tfrac{1}{4}(001/f, 110/(g,f)\,[0100])\,[0000].$$

The notation used in this form of the series (only) is indefinite since if a

---

* Stroh, *Mathematische Annalen*, Vol. 31 (1888), p. 444.

symbolical product C involves a large number of factors of the various types there will be several ways of forming the convolutions represented by $\zeta 00/C$ for instance. For this reason we shall usually represent the forms in the transvectants by their actual expressions as symbolical products (cf. (5)).

The degree of complexity to be anticipated in a reducing series is illustrated better by the following instance:

$$f = a_x^2\alpha_u^2 = b_x^2\beta_u^2, \quad g = a_a a_x \alpha_u = b_\beta b_x \beta_u,$$
$$(f,f)[1101] = a_\beta \alpha_u b_x (\alpha\beta x)(abu).$$

The transvectant,

$$T = (g, (f,f)[1101])[0110] = (c_\gamma c_x \gamma_u,\ a_\beta \alpha_u b_x (\alpha\beta x)(abu))[0110],$$

has the term,

$$t = a_\beta c_\gamma \alpha_u (\alpha\beta\gamma)(abu)(cbu).$$

By contraction with regard to $a_x^2\alpha_u^2$ it is shown that t is also a term of

$$S = -(f, (g,f)[0101])[1110] = (a_x^2\alpha_u^2,\ c_\gamma \beta_u b_x (\beta\gamma x)(cbu))[1110].$$

The series headed by T and S respectively are calculated by the methods shown in the previous problem. There are 17 terms in the resulting series, as follows:

$$\begin{aligned}
&(c_\gamma c_x \gamma_u,\ a_\beta \alpha_u (abu)(\alpha\beta x) b_x)[0110] - \tfrac{1}{2}(c_\gamma c_x \gamma_u,\ a_\beta \alpha_u (abu)(\{\alpha\beta\} bu))[1000] \\
&+ \tfrac{1}{2}(c_\gamma^2,\ a_\beta \alpha_u (abu)(\{\alpha\beta\} bu))[0000] - \tfrac{1}{6}(c_\gamma c_x \gamma_u,\ a_\beta b_a (abu)(\alpha\beta x))[0100] \\
&+ \tfrac{1}{6}(c_\gamma c_x \gamma_u,\ a_\beta \alpha_u b_x (\{\alpha\beta\} ab))[0100] \\
&= (a_x^2\alpha_u^2,\ c_\gamma \beta_u b_x (cbu)(\beta\gamma x))[1110] - \tfrac{1}{2}(a_x^2\alpha_u^2,\ c_\gamma b_x (\beta\gamma x)(\{bc\}\beta x))[0210] \\
&+ \tfrac{1}{6}(a_x^2\alpha_u^2,\ c_\gamma (\{bc\}\beta x)(\{\beta\gamma\} bu))[1100] + \tfrac{1}{6}(a_a a_x \alpha_u,\ c_\gamma b_x (\{(bc)\beta\}(\beta\gamma)u))[0100] \\
&+ \tfrac{1}{6}(a_x^2\alpha_u^2,\ c_\gamma b_x ((\beta\gamma)\{(bc)\beta\} u))[1100] - \tfrac{1}{2}(a_a a_x \alpha_u,\ c_\gamma \beta_u (bcu)(\{\beta\gamma\} bu))[1000] \\
&+ \tfrac{1}{6}(a_a a_x \alpha_u,\ c_\gamma (b\{\beta\gamma\} u)(\{bc\}\beta x))[0100] \\
&\qquad + \tfrac{1}{12}(a_a a_x \alpha_u,\ c_\gamma (bcu)(\beta\{(\beta\gamma) b\} x))[0100] \\
&- \tfrac{1}{2}(a_x^2\alpha_u^2,\ c_\gamma \beta_u (cbu)(\{\beta\gamma\} bu))[2000] \\
&\qquad + \tfrac{1}{12}(a_x^2\alpha_u^2,\ c_\gamma \beta_u (\{(\beta\gamma) b\}(cb) x))[1100] \\
&+ \tfrac{1}{12}(a_a a_x \alpha_u,\ c_\gamma \beta_u ((bc)\{(\beta\gamma) b\} x))[0100] \\
&\qquad + \tfrac{1}{12}(a_x^2\alpha_u^2,\ c_\gamma (cbu)(\{(\beta\gamma) b\}\beta x))[1100].
\end{aligned}$$

5. *The general series.* The series developed in the preceding section are special cases of something very general to which attention may now be given. Consider three connexes,

$$f = a_x^m \alpha_u^n, \quad g = b_x^\mu \beta_u^\nu, \quad h = c_x^\sigma \gamma_u^\tau.$$

The arbitrary transvectant of f and g is a monomial,

$$(f,g)[ijkl] = a_\beta^i \alpha_b^k (abu)^j (\alpha\beta x)^l a_x^\zeta b_x^\eta \alpha_u^\theta \beta_u^\iota,$$
$$\zeta = m - i - j, \quad \eta = \mu - j - k, \quad \theta = n - k - l, \quad \iota = \nu - i - l.$$

Suppose that p, q, r, s are fixed chosen integers which satisfy the conditions, (cf. (1)),

$$p+s \leqq j+\theta+\iota, \quad q+r \leqq l+\zeta+\eta, \quad p+q \leqq \sigma, \quad r+s \leqq \tau,$$

and consider any set of solutions of the linear diophantine equations,

$$(7) \quad \sigma_1+\sigma_2+\sigma_3 = p, \quad s_1+s_2+s_3 = q, \quad t_1+t_2+t_3 = r, \quad \tau_1+\tau_2+\tau_3 = s.$$

Under these conditions, Z being numerical,

$$(8) \quad T = (h, (f, g)[ijkl])[pqrs]$$
$$= \sum Z a_\beta{}^i \alpha_b{}^k (abc)^{\sigma_1} c_a{}^{\sigma_2} c_\beta{}^{\sigma_3} \; (c\{\alpha\beta\}u)^{s_1} (cau)^{s_2} (cbu)^{s_3}$$
$$\times (\gamma\{ab\}x)^{\tau_1} (\gamma\alpha x)^{\tau_2} (\gamma\beta x)^{\tau_3} \; (\alpha\beta\gamma)^{t_1} \gamma_a{}^{t_2} \gamma_b{}^{t_3} (abu)^{j-\sigma_1-\tau_1}$$
$$\times (\alpha\beta x)^{l-s_1-t_1} a_x{}^{\zeta-s_2-t_2} b_x{}^{\eta-s_3-t_3} \alpha_u{}^{\theta-\sigma_2-\tau_2} \beta_u{}^{\iota-\sigma_3-\tau_3} c_x{}^{\sigma-p-q} \gamma_u{}^{\tau-r-s}.$$

The terms of this summation are given by the totality of solutions in integers of the equations (7). For the term t from which we may develop a reducing series, we select the above summand, with $s_1 = \tau_1 = 0$, and contract t with regard to $a_x{}^m \alpha_u{}^n$. Hence,

$$t_{(1)} = \beta_v{}^i b_y{}^k (bcv)^{\sigma_1} c_y{}^{\sigma_2} (cuv)^{s_2} \; (\gamma xy)^{\tau_2} (\beta\gamma y)^{t_1} \gamma_v{}^{t_2}$$
$$\times (buv)^{j-\sigma_1} (\beta xy)^{l-t_1} (-1)^{s_2+\tau_2} F \cdot a_x{}^{\zeta-s_2-t_2} \alpha_u{}^{\theta-\sigma_2-\tau_2},$$
$$F = c_\beta{}^{\sigma_3} (cbu)^{s_3} \; (\gamma\beta x)^{\tau_3} \gamma_b{}^{t_3} b_x{}^{\eta-s_3-t_3} \beta_u{}^{\iota-\sigma_3-\tau_3} c_x{}^{\sigma-p-q} \gamma_u{}^{\tau-r-s}.$$

Therefore the term t of T is also a term of the transvectant,

$$(9) \quad S = (f, (-1)^\xi (h, g)[\sigma_3, s_3+\sigma_1, t_3, \tau_3+t_1])$$
$$\times [i+\sigma_1+t_2, j+s_2-\sigma_1, k+t_1+\sigma_2, l+\tau_2-t_1], \quad (\xi = \sigma_1+t_1+s_2+\tau_2).$$

Also since

$$(h, (f, g)[ijkl])[pqrs] = (-1)^{j+l} (h, (g, f)[kjil])[pqrs],$$

t is a term of

$$(10) \quad R = (g, (-1)^{\xi+j+l} (h, f)[\sigma_3, s_3+\sigma_1, t_3, \tau_3+t_1])$$
$$\times [k+\sigma_1+t_2, j+s_2-\sigma_1, i+t_1+\sigma_2, l+\tau_2-t_1].$$

Applying the methods of section 4, which were based on the theorem on the difference between a transvectant and one of its terms, we can immediately assert that the following reducing series exist, Γ, Δ, E being rational numbers:

$$(11) \quad T + \sum \mathrm{E} \, (00a/h, bcd/(f, g)[ijkl])[p_1q_1r_1s_1] u_x{}^{m_1}$$
$$= S + \sum \Delta \, (00\alpha/f, \beta\gamma\delta/(h, (-1)^\xi g)[\sigma_3, s_3+\sigma_1, t_3, \tau_3+t_1]) \; [p_2q_2r_2s_2] u_x{}^{m_2}$$
$$= R + \sum \Gamma \, (00\theta/g, \iota\kappa\lambda/(h, (-1)^{\xi+j+l} f)[\sigma_3, s_3+\sigma_1, t_3, \tau_3+t_1]) \; [p_3q_3r_3s_3] u_x{}^{m_3}.$$

These are finite series of multiple entry (cf. (6)), arranged according to descending grades of the transvectants. Every transvectant in a series is of lower grade than the one which forms the leading term of that series.

Let the grades of T, S, R be τ, σ, ρ, respectively. We refer to the set (T, S, R), with its grade numeral (τ, σ, ρ), as a reducing triad.

By analogous reasoning if we contract a monomial t of degree ν in the coefficients of ν forms $f_1, \cdots, f_\nu$, with regard to the symbolical letters which appertain to those respective forms, we obtain a reducing ν-ad of transvectants $(T_1, \cdots, T_\nu)$, whose grade numeral is of the form $(\tau_1, \cdots, \tau_\nu)$.

When we come to the applications of these series to the problem of reducing transvectants, where $f = g = h = a_x{}^m \alpha_u{}^n$, not more than one form of each reducing triad need be retained as irreducible, since every transvectant of the first summation is of grade lower than T and every transvectant in the second summation is of lower grade than S and the series is finite.

If we define the *index* * of $(P, Q)\,[ijkl]\,u_x{}^m$ to be the number,

$$e = 2i + j + 2k + l - 2m,$$

the terms of the sequences can be arranged according to both decreasing grade and decreasing index. The difference between a transvectant and one of its terms is exclusively a sum of terms of transvectants of both lower grade and lower index.

6. *Reduction of ordered sets.* The transvectants of a given degree derived from a connex $a_x{}^m \alpha_u{}^n$ can be arranged as a linear aggregate,

$$T_1, T_2, \cdots, T_i, \cdots, T_j, \cdots \quad (j > i), \tag{12}$$

in which the grade of T_j is $\geqq$ the grade of T_i. In searching for the irreducible transvectants we examine each term of (12) beginning with T_1, testing the cases, in order, for reducibility by separation, by identities, etc. If T_i is a transvectant T and S precedes T in (12) or is reducible in terms of those which precede or possesses a term which is so reducible, then T_i is reduced. In any event not more than one transvectant of the triad (T, S, R) need be retained. The number of irreducible cases for a given degree will be finite.

7. *Fundamental system of concomitants of $a_x{}^2 \alpha_u$ of the first three degrees.* The list (4) for this quantic is $f = a_x{}^2 \alpha_u$, $g = a_\alpha a_x$.

---

* *Transactions of the American Mathematical Society, loc. cit.*, p. 409. In the formula for e $2i$ represents the number by which the grade of PQ is reduced by i convolutions. The other terms of e have like interpretations when we attach to u_x the property of an operator which raises the grade by 2.

The transvectants of degree 2 are all monomials. Those which are not reducible by separation are arranged as an ordered set in the following table (to be read by rows):

$(g,g)[0100] = a_a b_\beta (abu)$, $(g,f)[1000] = a_a a_\beta b_x^2$, $(g,f)[0100] = a_a (abu)\beta_u b_x$,
$(f,f)[1000] = a_\beta \alpha_u a_x b_x^2$, $(f,f)[0100] = (abu)\alpha_u \beta_u a_x b_x$, $(f,f)[0200] = (abu)^2 \alpha_u \beta_u$,
$(f,f)[1100] = a_\beta (abu)\alpha_u b_x$, $(f,f)[0010] = b_a \beta_u a_x^2 b_x$, $(f,f)[0001] = (\alpha\beta x) a_x^2 b_x^2$,
$(f,f)[1010] = a_\beta b_a a_x b_x$, $(f,f)[0101] = (abu)(\alpha\beta x) a_x b_x$, $(f,f)[0201] = (abu)^2 (\alpha\beta x)$,
$(f,f)[0110] = b_a (abu)\beta_u a_x$, $(f,f)[1110] = a_\beta b_a (abu)$.

Referring to these quantics by their numbers in the set we note that a change of sign occurs in (1), (5), (9), (12), (14) when the pairs (a,α), (b,β) are interchanged. Hence these forms vanish. Likewise (8)=(4), (13)=(7). We have

$$(13) \qquad (f,f)[0101] = \begin{vmatrix} a_a & a_\beta & a_x \\ b_a & b_\beta & b_x \\ \alpha_u & \beta_u & u_x \end{vmatrix} a_x b_x = g^2 u_x + 2(4) - 2fg - (10)u_x.$$

Hence (11) is reducible. There remain six concomitants, (2), (3), (4), (6), (7), (10), for which we use the notation, *A, B, C, D, E, F.*

The next table contains the ordered set of non-separable transvectants of degree 3. To obtain their explicit symbolical expressions we represent the first quantic in each as $f = c_x^2 \gamma_u$, or $g = c_\gamma c_x$, and the second quantic in the form given in the above list for the degree 2. In accordance with the theory it is sufficient in most developments to write one term only of each transvectant. Thus, $(f,F)[0110] = a_\beta a_\gamma b_a (cbu) c_x$.

$(g,A)[0100]$, $(g,F)[0100]$, $(g,B)[0100]$, $(g,B)[1000]$, $(g,E)[0100]$, $(g,E)[1000]$,
$(g,C)[0100]$, $(g,C)[1000]$, $(g,D)[1000]$, $(f,g^2)[0200]$, $(f,g^2)[0110]$, $(f,A)[0100]$,
$(f,A)[0200]$, $(f,A)[0010]$, $(f,A)[0110]$, $(f,F)[0100]$, $(f,F)[0200]$, $(f,F)[0010]$,
$(f,F)[0110]$, $(f,B)[0010]$, $(f,B)[1010]$, $(f,B)[2010]$, $(f,B)[0001]$, $(f,B)[1001]$,
$(f,B)[0101]$, $(f,B)[1101]$, $(f,B)[1000]$, $(f,B)[2000]$, $(f,B)[1100]$, $(f,B)[0100]$,
$(f,E)[0001]$, $(f,E)[1001]$, $(f,E)[0101]$, $(f,E)[1101]$, $(f,E)[0010]$, $(f,E)[1010]$,
$(f,E)[2010]$, $(f,E)[1000]$, $(f,E)[2000]$, $(f,E)[0100]$, $(f,E)[1100]$, $(f,gf)[0210]$,
$(f,C)[0001]$, $(f,C)[0101]$, $(f,C)[0201]$, $(f,C)[0010]$, $(f,C)[1010]$, $(f,C)[0110]$,
$(f,C)[1110]$, $(f,C)[0210]$, $(f,C)[1000]$, $(f,C)[1100]$, $(f,C)[0100]$, $(f,C)[0200]$,
$(f,D)[1000]$, $(f,D)[2000]$, $(f,D)[0001]$, $(f,D)[1001]$, $(f,D)[2001]$, $(f,f^2)[1001]$,
$(f,f^2)[1101]$, $(f,f^2)[2000]$, $(f,f^2)[2010]$, $(f,f^2)[0210]$.

The actual operation of reducing series in the problem of the discovery of all reducible transvectants in this set will now be shown in detail in a score of cases. The series (6), of which (11) is the general expression, give all

reductions in this problem, including those types for which Clebsch and Gordan employed a special identity. The number of irreducible transvectants is 24. These are collected, finally, in explicit symbolical form, in a separate list, one term, only, of each transvectant being written.

(a) Consider the case,

$$S = (f, -B)\,[1010] = -(f, (g, f)\,[0100])\,[1010] = a_\alpha c_\beta (abc)\gamma_u b_x.$$

Contracting t with regard to $a_x^2\alpha_u$ we find it to be a term, also, of

$$T = (g, -E)\,[1000] = -(g, (f, f)\,[1100])\,[1000].$$

Likewise t is a term of $(f, B)\,[2000] = R$ on the basis of contraction with regard to $c_x^2\gamma_u$. The first series of (11) is that given in (6). The reducing triad is

$$\{T, S, R\}, \tag{14}$$

with the grade numeral $(4, 6, 6)$. Hence S, R are reducible in terms of T and other transvectants, $T_1, T_2, T_3, S_1, S_2, \cdots$ of grade < 6: The calculation does not reduce T, in fact it is ultimately retained in the irreducible system.

The transvectants of the aggregate being ordered it is unnecessary to compute each reducing sequence in the explicit form (6). It is sufficient to determine the reducing triad by contraction except in some cases where one or more members of the triad vanish identically and in certain other cases where the methods are augmented by special processes.

(b) $(g, E)\,[0100] = a_\beta c_\gamma (abu)\,(cbu)\,\alpha_u$. Contraction with regard to $a_x^2\alpha_u$ and $b_x^2\beta_u$, respectively, gives the reducing triad,

$$\{(g, E)\,[0100],\ (f, B)\,[1100],\ (f, fg)\,[0210]\},$$

for which the grade numeral is $(4, 6, 7)$. The first form is not reduced but when the grades 6, 7 are reached in the problem of the discovery of the irreducible transvectants of the aggregate the other two forms of the triad are reduced in terms of $(g, E)\,[0100]$ and other transvectants of grade less than that of the transvectant considered.

(c) $(f, A)\,[0200] = a_\alpha a_\beta (cbu)^2\gamma_u$. The reducing triad is

$$\{(f, A)\,[0200],\ (g, D)\,[1000],\ (f, fg)\,[0210]\},$$

with the grade numeral $(5, 5, 7)$. We retain the second transvectant. Instead, we could retain the first.

(d) $(f, A)[0100] = a_a a_\beta (cbu)\gamma_u b_x c_x$. Contracting with reference to $a_x^2 \alpha_u$ we find

$$S = (a_a a_x,\ (cbu)\beta_u \gamma_u b_x c_x)[1000] = 0.$$

The reducing triad is

$$\{(f, A)[0100],\ 0,\ -(f, fg)[0110]\},$$

with the grade numeral $(5, 6, 7)$. We resort to a special process. Since,

$$S = \tfrac{1}{2}\{a_a (acb)\beta_u \gamma_u b_x c_x + t + a_a a_\gamma (cbu)\beta_u b_x c_x\},$$

in which the first term is zero, there results

$$S - t = a_a (cbu)(\{\gamma\beta\}au) b_x c_x.$$

Contracting the latter with regard to $a_x^2 \alpha_u$ it is found to be a term of

$$(a_a a_x,\ (\beta\gamma x)(bcu) b_x c_x)[0100] = (g, g^2 u_x + 2C - 2gf - Fu_x)[0100].$$

Now $(g, C)[0100]$, of grade 5, is one of the forms retained as irreducible while $(g, g^2)[0100]$, $(g, gf)[0100]$, $(g, F)[0100]$ are lower forms in the problem of the grade 5. Hence t is reduced. No form is retained from the reducing triad.

(e) $(f, F)[0110] = a_\beta a_\gamma b_a (cbu) c_x$. The reducing triad is

$$\{(f, F)[0110],\ (f, (cbu)\beta_u \gamma_u b_x c_x)[2010],\ -(f, C)[1110]\},$$

for which the grade number is $(5, 8, 7)$. The second form is zero but the interval from 5 to 8 for the grades is too large for successful use of the method employed in (d). The first form is retained and the last one is reduced.

(f) $(f, E)[2010] = a_\beta b_\gamma c_a (abc)$. This transvectant is reproduced by contraction in any one of the three possible ways. Hence the three reducing series (11) are identical and there is no reduction. In fact this is the only pure invariant * of degree 3 and the invariant of lowest degree of f.

(g) $(f, E)[1101] = a_\beta (abc)(\gamma\alpha x)(cbu)$. We find the reducing triad,

$$\{(f, E)[1101],\ (f, -D)[2001],\ (f, C)[1110]\},$$

with the grade numeral $(6, 7, 7)$. The first form is not reduced by this.

Expansion gives

* There is an invariant of degree 6, $(dea)(def) c_a c_\gamma a_\delta b_\varepsilon b_\zeta f_\beta$.

$$t = \begin{vmatrix} a_\gamma & a_a & a_x \\ b_\gamma & b_a & b_x \\ c_\gamma & c_a & c_x \end{vmatrix} \quad (cbu)a_\beta = a_\beta(cbu)a_\gamma b_a c_x + a_\beta(cbu)b_\gamma c_a a_x$$

$$+ a_\beta(cbu)c_\gamma a_a b_x - a_\beta(cbu)c_\gamma b_a a_x - a_\beta(cbu)a_\gamma c_a b_x - a_\beta(cbu)b_\gamma a_a c_x.$$

By contraction we find these six monomials to be terms of the respective transvectants,

$(f, F)\,[0110]$, $(f, E)\,[1010]$, $(g, B)\,[1000]$,
$(g, F)\,[0100]$, $(f, F)\,[0110]$, $(g, E)\,[1000]$.

These are all of grade < 6 excepting the second and, since it is of grade 6 and is one of the forms retained as irreducible, $(f, E)\,[1101]$ and, therefore, each form of the reducing triad, is reduced.

The calculations which were necessary for the preparation of the following table,* involved the reduction of over fifty transvectants.

TABLE I.

The Complete System of f of the First Three Degrees.

Degree 1.

$$f = a_x^2 \alpha_u, \quad g = a_a a_x$$

Degree 2.

$$(g, f)\,[1000] = a_a a_\beta b_x^2, \quad (g, f)\,[0100] = a_a(abu)\beta_u b_x$$
$$(f, f)\,[1000] = a_\beta \alpha_u a_x b_x^2, \quad (f, f)\,[0200] = (abu)^2 \alpha_u \beta_u$$
$$(f, f)\,[1100] = a_\beta(abu)\alpha_u b_x, \quad (f, f)\,[1010] = a_\beta b_a a_x b_x$$

Degree 3.

$(g, A)\,[0100] = a_a a_\beta c_\gamma (cbu) b_x$,
$(g, B)\,[0100] = a_a c_\gamma (abu)(cbu)\beta_u$,
$(g, E)\,[1000] = a_\beta c_a c_\gamma (abu) b_x$,
$(g, D)\,[1000] = c_a c_\gamma (abu)^2 \beta_u$,
$(f, F)\,[0100] = a_\beta b_a (cbu)\gamma_u a_x c_x$,
$(f, F)\,[0010] = a_\beta a_\gamma b_a b_x c_x^2$,
$(f, E)\,[0001] = a_\beta (abu)(\gamma \alpha x) b_x c_x^2$,
$(f, E)\,[1010] = a_\beta b_\gamma c_a (abu) c_x$,
$(f, E)\,[1000] = a_\beta (abc)\alpha_u \gamma_u b_x c_x$,
$(f, E)\,[1100] = a_\beta c_a (abu)(cbu)\gamma_u$,
$(f, C)\,[0010] = a_\beta a_\gamma \alpha_u b_x^2 c_x^2$,
$(f, C)\,[0200] = a_\beta (cbu)^2 \alpha_u \gamma_u a_x$,

$(g, F)\,[0100] = a_\beta b_a c_\gamma (cau) b_x$
$(g, E)\,[0100] = a_\beta c_\gamma (abu)(cbu)\alpha_u$
$(g, C)\,[0100] = a_\beta c_\gamma (cau)\alpha_u b_x^2$
$(f, A)\,[0010] = a_a a_\beta b_\gamma b_x c_x^2$
$(f, F)\,[0200] = a_\beta b_a (cau)(cbu)\gamma_u$
$(f, F)\,[0110] = a_\beta a_\gamma b_a (cbu) c_x$
$(f, E)\,[0010] = a_\beta b_\gamma (abu)\alpha_u c_x^2$
$(f, E)\,[2010] = a_\beta b_\gamma c_a (abc)$
$(f, E)\,[0100] = a_\beta (abu)(cbu)\alpha_u \gamma_u c_x$
$(f, C)\,[0001] = a_\beta (\gamma \alpha x) a_x b_x^2 c_x^2$
$(f, C)\,[0100] = a_\beta (cau)\alpha_u \gamma_u b_x^2 c_x$
$(f, D)\,[2000] = (abc)^2 \alpha_u \beta_u \gamma_u$

* The form u_x of zero degree belongs to the list.

8. *Theory of symbolic moduli.* Simplification and further condensation of notations in connection with the processes which relate to the discovery of complete systems is attained by means of a theory of moduli.

Definitions. If a system (A) of connexes is finite and complete and (B) is finite and complete then the system (C) which consists of all terms of all transvectants $(A, B)[ijkl]$ is also finite and complete (*Transactions of the American Mathematical Society, loc. cit.,* 411).

A system such as (C) is said to be relatively finite with respect to a set of constant symbolic moduli a_α, $a_\alpha a_\beta$, (abc), $\cdots$, when every form of (C) is expressible as a polynomial in a finite set of forms (C) together with forms which contain at least one of the moduli as a factor.

A multiple convolution in a product N of powers of forms (A) may be a polynomial. We express this by $/N = \Sigma//N$ and each form $//N$ may be obtained singly by the properly chosen convolution $\zeta\eta\theta/N$.

A system, as (C), is said to be relatively complete with regard to a set of moduli when every form $//N$ obtained by convolution from a product of forms (C) is a polynomial in forms (C) together with terms containing a modulus as a factor.

Special Lemma 1. *The two forms* $f = a_x^2\alpha_u$, $D = (abu)^2\alpha_u\beta_u$, *of Table I constitute a system which is relatively finite and complete modulis* a_α, b_γ, $(abc)^2$.

Since the forms are finite in number we need only to investigate the forms obtainable by the processes of convolution between products.

No process of convolution is applicable to g. From $D^q f^r$ we obtain only expressions $D^s f^t$ modulis a_α, a_β, (abc), $(\alpha\beta\gamma)$, as is evident from Table I together with the following considerations relating to convolutions from D^q. From D alone we obtain,

$$010/D = (abu)^2(\alpha\beta x) = 0,$$
$$010/D = (abu)(\{ab\}\alpha x)\beta_u = (abu)(\{ab\}\beta x)\alpha_u = B + E,$$
$$020/D = (\{ab\}\alpha x)(\{ab\}\beta x) = 2A - g^2 - F.$$
$$(D, D)[000l] = \Sigma(abu)(\alpha\gamma x)P + \Sigma(\{ab\}\tau x)Q, \quad (\tau = (cd),\ \gamma \text{ or } \delta).$$

All terms of both summations contain factors a_α, b_γ or a_τ.

We next prove that (abc) is a *reducible modulus,* that is, any monomial which contains it can be transformed into a linear expression in monomials each containing a factor a_α or b_γ or $(abc)^2$.

If any monomial $t = (abc)\,P$ be contracted with regard to the three pairs (a, α), (b, β), (c, γ) it is seen to be a term of,

$$T = (/Q, R)\,[ijkl],$$

where R does not contain the symbols of these pairs and Q is the leader of the form-sequence * which appertains to (abc), viz., $Q = (abc)a_x b_x c_x \alpha_u \beta_u \gamma_u = 0$. Hence

$$t = \Sigma I(/Q, /R)\,[ijkl]\,u_x^m, \tag{15}$$

I being numerical. All transvectants of the summation in which no actual convolution occurs in Q vanish. All convolutions in Q result in factors a_α, b_γ excepting $G = (abc)^2 \alpha_u \beta_u \gamma_u$ and three instances equivalent, by interchanges of symbols, to

$$H = /Q = (abc)\,(abu)\,\alpha_u \beta_u \gamma_u c_x.$$

But,

$$H - c_\alpha (abu)^2\,\beta_u \gamma_u c_x = a_\alpha (abu)\,(bcu)\,\beta_u \gamma_u c_x - b_\alpha (abu)\,(acu)\,\beta_u \gamma_u c_x.$$

A like argument deletes the factor $(\alpha\beta\gamma)$ from the modular system. The lemma is proved.

COROLLARY. A monomial which contains a factor $a_\alpha(abc)$ can be transformed into a sum each term of which contains a constant composed of two factors chosen from the types a_α, b_γ.

Special Lemma 2. The quantics

$$g = a_\alpha a_x,\ C = a_\beta a_x b_x^2 \alpha_u,\ E = a_\beta (abu) \alpha_u b_x,\ G = (abc)^2 \alpha_u \beta_u \gamma_u,$$

each containing one and only one of the moduli a_α, b_γ, $(abc)^2$ as a factor, constitute a system which is relatively finite and complete modulis:

$$(S):\ a_\alpha a_\beta,\ a_\beta b_\alpha,\ a_\alpha b_\gamma,\ a_\beta c_\delta,\ a_\alpha b_\beta,\ a_\beta (cde)^2,\ (abc)^2 (\ddot{a}\ddot{e}\ddot{f})^2.$$

To prove this it is only necessary to identify the results of convolution in the list which follows, from which convolutions which vanish identically have been omitted. Any convolution in gG contains $a_\alpha a_\beta$. A convolution in any product of powers of g, C, E, G, other than gG, contains one of the last five moduli as a factor.

$$\begin{aligned} &100/C = a_\beta (abu) \alpha_u b_x = E,\quad 001/C = a_\beta b_\alpha a_x b_x = F, \\ &001/C = a_\alpha a_\beta b_x^2 = A, \qquad 110/C = 010/E = a_\beta (\{ab\}\alpha x) b_x = A - F. \end{aligned} \tag{16}$$

* Cf. § 9.

9. *The modular form-sequence* (*Formenreihe, Noether*). A quantic which has a given constant symbolic factor, for example $P = a_\alpha b_\gamma (abu)\beta_u c_x^2$, is a member of a sequence of forms which are obtainable from a given one of the sequence, called the leading form, and which all have this same constant as their only constant factor. We use only *outer* indices j, l of the matrix $[ijkl]$ in constructing a form-sequence and omit convolutions which would bring an additional constant factor into a form. The form-sequence of which P is a term is,

$$(17) \qquad L = a_\alpha b_\gamma a_x b_x c_x^2 \beta_u,\ P,\ a_\alpha b_\gamma (acu)\beta_u b_x c_x,\ a_\alpha b_\gamma (bcu)\beta_u a_x c_x,\ a_\alpha b_\gamma (acu)(bcu)\beta_u.$$

The form $a_\alpha b_\gamma (\{ab\}\beta x) c_x^2$, for instance, is not included since it equals $a_\alpha b_\gamma a_\beta b_x c_x^2 - a_\alpha b_\gamma b_\beta a_x c_x^2$.

The rank of a modulus. The factors of a modulus are of three types, a_r, (abc), $(\alpha\beta\gamma)$. However it is usual to write $(abc) = a_{(bc)} = a_s$, $(\alpha\beta\gamma) = t_\alpha$, thus reducing all types to one. The number of factors of the type a_s in a modulus,

$$M = a_s{}^\rho b_t{}^\sigma \cdots n_w{}^\tau,$$

is the rank of M. Thus the modular systems used in the special lemmas 1, 2 are not homogeneous in rank.

General Lemma. If a finite system (A) *of connexes,* $A_1, A_2, \cdots$, *all of which are concomitants of* $f = a_x{}^m \alpha_u{}^n$, *includes* f *and is relatively complete for a set* (G) *of constant symbolic moduli,* $G_1, G_2, \cdots$, *and if, also a finite system* (B) *of connexes* $B_1, B_2, \cdots$; *concomitants of* f, *is relatively complete for a set* (Γ) *of moduli* $\Gamma_1, \Gamma_2, \cdots$, *and contains the leading form of the form-sequence whose quantics have as their only constant symbolic factor the modulus* G_i, *for each case* $i = 1, 2, \cdots$, *then the system* (C) *derived by transvection from* (A) *and* (B) *is relatively finite and complete, modulo* (Γ).

The proof is in two parts.

Proposition I. If $//P$ is an expression derived by convolution from a power P of f, any term t of $\sigma = (//P, B)[ijkl]$ can be expressed (mod (Γ)) as an aggregate of transvectants of the type $\tau = (A, B)[ijkl]$ where A, B are forms from the respective systems (A), (B) and A is of a degree $\leqq$ the degree of P.

The proof is by induction with reference to advancing degrees of P, initiated by the case ($k \neq 0$, a numerical constant), $(ku_x{}^m, B)[0000]$ where $//P$ is of zero degree and the proposition is evidently true.

Assume the proposition when the degree of $//P$ is $< r$. If $//P$ is next assumed of degree r we have (cf. (3)),

$$t = (//P, B)[ijkl] + \Sigma I(///P, /B)[i_1j_1k_1l_1]u_x^{m_1},$$

the transvectants under Σ being of lower grade and index than σ. Also (Hyp.),

$$//P = F(A) + \sum_i G_iY_i \equiv F(A) \pmod{(G)},$$

$$(18) \qquad ///P = F'(A) + \sum_j G_jY_j' \equiv F'(A) \pmod{(G)},$$

$$/B = \Phi(B) + \sum_k \Gamma_kZ_k \equiv \Phi(B) \pmod{(\Gamma)}.$$

Hence the expression for t contains,

(a) Transvectants of the type $(F(A), \Phi(B))[ijkl]$, the degree of $F(A)$ being r.

(b) Transvectants $(G_iY_i, B)[i'j'k'l']$ where the degree of G_iY_i is r.

(c) Transvectants whose terms are zero (mod Γ).

In the cases (a), (c) the proposition is obvious. For (b) note that if we contract G_iY_i with regard to all of the symbols in G_i, and these only, it is seen to be a term of $(/B_i, X_i)[pqrs]$ where B_i is the leader of the form-sequence corresponding to G_i, (multiplied, at most, by a power of f). We can now contract any term s of $(G_iY_i, B)[i'j'k'l']$, with reference to the symbols in B_iB. Hence s is a term of $(P_1, //B_iB)[ijkl]$ and, (cf. (9)),

$$s = (P_1, //B_iB)[ijkl] + \Sigma I(//P_1, \quad ///B_iB)[i_1j_1k_1l_1]u_x^{m_1},$$

in which $P_1, //P_1$ are of degree $< r$. Since

$$//B_iB \equiv \Phi_1(B) \pmod{(\Gamma)},$$

the proposition is proved true also in the case (b), the induction being complete.

Proposition II. The system (C) is relatively finite and complete modulo (Γ).

We arrange the transvectants of (C) as an ordered aggregate according to three rules, viz.,

(a) The transvectants of (C), viz., $(A, B)[ijkl] = \tau$, are arranged according to ascending degrees of AB.

(b) Those for which the degrees of AB are the same are arranged according to ascending degrees of A.

(c) Transvectants of like degrees for both AB and A are arranged according to ascending grade.

Further than this the ordering is immaterial.

Let t, t' be two non-vanishing terms of τ. Then

$$t' - t = \sum I(/A, /B)\,[i_1 j_1 k_1 l_1]\,u_x^m,$$

and every transvectant under $\sum$ is of lower grade than τ. From (18) (Hyp.),

$$t' - t = \sum (F(A), \Phi(B))\,[i_1 j_1 k_1 l_1]\,u_x^{m_1} + \sum (G_i Y_i, \Phi(B))\,[i_2 j_2 k_2 l_2]\,u_x^{m_2} \pmod{(\Gamma)}.$$

The transvectants under the first summation belong to the ordered aggregate and are of lower grade than τ. The degree of $F(A)$ is equal to that of A. By the argument of the proof of Proposition I the second $\sum$ is expressible in the form,

$$\sum (A', B')\,[pqrs]\,u_x^n,$$

where A' is of lower degree than A.

It is now plain that if all terms of all transvectants coming before τ in the ordered aggregate are polynomials in a set,

$$C_1, C_2, \cdots, C_a, u_x,$$

except for terms which involve a modulus Γ_k as a factor, then, all terms of transvectants up to and including τ can be expressed (mod (Γ)) as polynomials in the set,

$$C_1, C_2, \cdots, C_a, t, u_x.$$

Also if τ has a term t which is reducible (mod (Γ)) in terms of $C_1, \cdots, C_a, u_x$, then τ is completely reduced without the adjunction of t to the latter set. Thus, in building up the system of C's in terms of which all forms (C) are rationally and integrally expressible modulo (Γ), by proceeding from one transvectant τ to the next one in the aggregate, we adjoin a new form to the set of irreducible ones already identified only when we find a transvectant which has only irreducible terms t. But the number of such irreducible transvectants in (C) is finite. Hence (C) is relatively finite modulo (Γ).

The fundamental system (mod (Γ)) may be chosen by taking one term from each transvectant which contains no reducible term or by choosing the irreducible transvectants themselves, (cf. (3)).

To prove that (C) is relatively complete modulo (Γ), note that any term $//X$ derived by convolution from

$$X = C_1^{\gamma_1} C_2^{\gamma_2} \cdots C_q^{\gamma_q},$$

is a term of a transvectant $(//A, //B)[ijkl]$ where $//A$, $//B$ are derived by convolution from forms (A), (B) respectively. Hence,

$$//X = (//A, //B)[ijkl] + \Sigma I(///A, ///B)[i_1 j_1 k_1 l_1] u_x^m$$
$$\equiv \Sigma (A', B')[pqrs] u_x^n \pmod{(\Gamma)},$$

where A' is a form (A) of degree $\leqq$ the degree of $//A$. All transvectants of the latter summation are polynomials in a finite set,

$$C_1, C_2, \cdots, C_q, u_x.$$

Hence the system (C) is relatively complete, as well as finite, modulo (Γ).

10. *The special investigations resumed.* The general Lemma, used in connection with the methods relating to the special lemmas 1, 2, enables us to construct systems of concomitants which are relatively finite and complete with respect to a set of moduli of rank $\geqq \nu$ from a system which is relatively finite and complete with respect to a set of rank $\geqq \nu - 1$. Since the number of such forward steps is finite * an absolutely finite and complete system will be reached. For example the system derived by transvection from $\{f, D\}$ and $\{g, C, E, G\}$ is relatively finite and complete modulo (S). We can also increase the second set g, C, E, G by adjunction of other forms, as A and F, making corresponding alterations in the set of moduli.

An outstanding problem is how to reduce a modular system to the minimum number. In the binary theory, due to Gordan, there is one modulus in each system of moduli, that for the rank $2k$ being $(ab)^{2k}$ and that for the next higher rank, $(2k+2)$, being $(ab)^{2k+2}$. The theory underlying the binary transformations is a lemma due to Jordan: If $v_1 + v_2 + v_3 = 0$, then any product of powers of v_1, v_2, v_3, of order n can be expressed linearly in terms of such products as contain an exponent $\geqq \frac{2}{3}n$. For use in the binary theory the equality is,

$$(ab)c_x + (bc)a_x + (ca)b_x = 0.$$

The generalization of Jordan's lemma to four variables has been made by Wood and by A. Young.† If

$$v_1 + v_2 + v_3 + v_4 = 0,$$

for example,

---

* Grace and Young, *loc. cit.*, p. 274.

† *Proceedings of the London Mathematical Society*, Series 2, Vol. 1 (1901), p. 345; Vol. 22 (1923), p. 185.

$$(19) \qquad (abc)d_x - (bcd)a_x + (cda)b_x - (dab)c_x = 0,$$

any monomial of degree n in v_1, v_2, v_3, v_4 can be expressed linearly in terms of such as contain an exponent $\geqq \frac{1}{2}n$. Young's paper deals with perpetuants which are free from u_1, u_2, u_3. No extensions of Jordan's lemma which would be applicable to mixed concomitants have been made.

Modular theory of the system of $f = a_x\alpha_u$. The form f constitutes a system which is relatively finite and complete modulis a_α, a_β. The form $f_1 = b_\alpha a_x b_u$ constitutes a system which is relatively finite and complete modulis $a_\beta b_\alpha\ (= i_1), b_\alpha d_\gamma$. Since $i = a_\alpha$ is an invariant and f_1 contains as a factor the modulus $a_\beta\ (= b_\alpha)$, the General Lemma states that the transvectant system,

$$(20) \qquad (f^r, f_1^s)[ijkl],$$

with i, is relatively finite and complete modulis $i_1, b_\alpha d_\gamma$. At this point it was necessary to test all transvectants (20), not reducible by separation, for reducibility as members of reducing ν-ads. The irreducible cases are i_2, g, h,

$$i_2 = a_\beta b_\gamma c_\alpha, \quad g = b_\alpha(\beta\gamma x)a_x c_x, \quad h = b_\alpha(acu)\beta_u\gamma_u.$$

The modulus $b_\alpha d_\gamma$ contains four symbols. Any form which contains it as a factor is of a degree, in the coefficients of f, $\geqq 4$ and hence is reducible in terms of the irreducible forms already found. Thus $b_\alpha d_\gamma$ is a reducible modulus. Since i_1 is an invariant the following system * is absolutely complete:

$$u_x, i, i_1, i_2, f, f_1, g, h.$$

11. *The relatively finite and complete system of* $f = a_x{}^2\alpha_u$, (mod (S)). The set of symbolic moduli (Γ) of the second rank for $a_x{}^2\alpha_u$ is the set (S) of Special Lemma 1. The transvectants,

$$(21) \qquad (f^n D^p, g^r C^s E^t G^u)[ijkl],$$

which are not reducible, form a relatively finite † and complete system (mod (S)). Hence we first determine the non-separable transvectants of this formula, (cf. Table II). The biternary reducing series (11), while still useful in performing reductions, is not infallible as in the problem of investigating from degree to degree for the irreducible concomitants. For, a reducing ν-ad will here usually contain transvectants which do not occur in the system (21). An example is the triad,

---

* Due to Clebsch and Gordan, *Mathematische Annalen*, Vol. 1 (1869), p. 359.

† E. Noether, *Journal für Mathematik*, Vol. 134 (1908), p. 90.

$$\{(f, E)[0010],\ (f, C)[1100],\ (-f, f^2)[1110]\}.$$

This is analogous to the corresponding situation in the binary theory. The number of non-separable transvectants of the formula (21) exceeds 250. The following list includes all which have no modulus (S) as a factor, omitting a small number which were reduced (mod (S)) by the biternary series. In constructing the table we write all pairs of products of quantics for which (21) is of a given grade > 3. The totality of matrices $[ijkl]$ which may be used with a given pair of products is constructed from the integers which satisfy (1). It is then an elementary exercise in tactic to identify the cases which are not separable. An example of reduction by series is afforded by the pair,

$$T = (f, C)[1100], \quad S = (f, E)[0010],$$

which were shown above to be terms of the same triad. From (11),

$$T + \Sigma\, \epsilon(/f, /C)[ijkl]\, u_x^m = S + \Sigma\, \delta(/f, /E)[pqrs]\, u_x^n,$$

and from this formula there follows a relation (cf. (16)),

$$T \equiv S + \Sigma\, I(f, E)[abcd]\, u_x^e \pmod{(S)}.$$

If one term, only, of each transvectant (cf. Table I) is written in the table the equalities are congruencial modulis transvectants of this same list, of lower grade, of forms derived from the given forms by convolution.

THEOREM. *The irreducible transvectants of Table II constitute a system for $a_x^2 \alpha_u$, which is relatively finite and complete modulo (S), when the forms f, D, g, C, E, G, u_x are added thereto.*

TABLE II.

Grade 4. $(f, g)[0010]$, $(f, g)[0100]$. *Grade* 5. $(D, g)[0010]$

Grade 6.

$(f, E)[0001]$, $(f, E)[1001]$, $(f, E)[0101]$, $(f, E)[1101]$, $(f, E)[0010]$
$(f, E)[1010]$, $(f, E)[2010]$, $(f, E)[1000]$, $(f, E)[0100]$, $(f, E)[1100]$
$(f, G)[1000]$, $(f, G)[2000]$, $(f, G)[1001]$, $(f, G)[0001]$, $(f, G)[2001]$

Grade 7.

$(f, C)[0001]$, $(f, C)[0101]$, $(f, C)[0201]$, $(f, C)[0010]$, $(f, C)[1010]$
$(f, C)[0100]$, $(f, C)[0200]$, $(D, E)[0010]$, $(D, E)[0001]$, $(D, E)[0002]$
$(D, E)[0011]$, $(D, E)[0012]$, $(D, G)[0001]$, $(D, G)[0002]$, $(D, G)[0003]$
$(f, gG)[1100]$, $(f, gG)[1010]$, $(f, gG)[0101]$, $(f, gG)[1101]$, $(f, gG)[2010]$

Grade 8.

$(D, C)[0010]$, $(D, C)[0001]$, $(D, C)[0020]$, $(D, C)[0021]$, $(D, C)[0011]$
$(D, C)[0030]$, $(D, C)[0031)$, $(D, gG)[0011]$, $(D, gG)[0012]$, $(D, gG)[0013]$

Grade 9.

$(f^2, E)[0011]$, $(f^2, E)[1011]$, $(f^2, E)[0002]$, $(f^2, E)[0102]$, $(f^2, G)[3000]$
$(f^2, G)[0002]$, $(f^2, G)[1002]$

Grade 10.

$(f^2, C)[0020]$, $(f^2, C)[1020]$, $(f^2, C)[0120]$, $(f^2, C)[0300]$, $(f^2, C)[1120]$
$(f^2, C)[0011]$, $(f^2, C)[0111]$, $(f^2, C)[0211]$, $(f^2, C)[0301]$, $(f^2, C)[1200]$
$(f^2, C)[1210]$, $(f^2, C)[1300]$, $(fD, E)[1011]$, $(fD, E)[0102]$, $(f^2, gG)[3100]$
$(f^2, gG)[3010]$, $(f^2, gG)[0102]$, $(f^2, gG)[1102]$, $(fD, G)[1002]$

Grade 11.

$(fD,C)[1020]$, $(fD,C)[0120]$, $(fD,C)[1030]$, $(fD,C)[0111]$, $(fD,C)[1120]$
$(fD,C)[0121]$, $(fD,C)[0211]$, $(fD,gG)[1102]$

Grade 12. $(f^3, E)[0012]$, $(f^3, G)[0003]$. *Grade* 16. $(f^4, C)[0031]$

Grade 13.

$(f^3, C)[0030]$, $(f^3, C)[1030]$, $(f^3, C)[0021]$, $(f^3, C)[0121]$, $(f^3, gG)[0103]$.

12. *Conclusion.* We add some data concerning the modular systems of rank $\geqq 3$ and the next step of the general method of finding an absolutely complete system for $a_x{}^2\alpha_u$. Some progress can be made toward reducing a modulus such as $a_\beta c_\delta$ in terms of other products by use of the jacobian identity (19). But, in view of the extent of Table II it is obvious that the next transvectant system will yield several thousand non-separable cases. We can encompass this system by theoretical considerations. We readily find forms which have the respective moduli of (S) as their only constant factors, as follows: $a_\alpha b_\beta : g^2$; $a_\alpha a_\beta : A$; $a_\beta b_\alpha : F$; $a_\alpha b_\gamma$: The form-sequence (17) led by $L = gC$; $a_\beta c_\delta$: The form-sequence $(B_1, \cdots, B_{10})$ led by $B_1 = C^2$; $a_\beta (cde)^2$: The form-sequence of CG; $(abc)^2(def)^2$: The form-sequence led by G^2. If these quantics be written $B_1, \cdots, B_\tau$, and those of Table II as $A_1, \cdots, A_\sigma$, the irreducible transvectants of the formula,

$$(22) \qquad (A_1{}^{a_1} \cdots A_\sigma{}^{a_\sigma}, B_1{}^{b_1} \cdots B_\tau{}^{b_\tau})[ijkl],$$

compose a system which is relatively finite and complete (mod (Γ)), (Γ) being the set of moduli with respect to which the system $B_1, \cdots, B_\tau$ is relatively finite and complete. The set (Γ) is of rank $\geqq 3$. If it should consist of pure invariants only, the formula (22) would give the fundamental system.

THE UNIVERSITY OF PENNSYLVANIA.

Transitive Substitution Groups Containing Regular Subgroups of Lower Degree.

By F. E. Johnston.

§ I. *Introduction.*

In 1831 E. Galois noted * that in any substitution group the totality of substitutions each of which leaves invariant a particular letter constitute a subgroup. This subgroup, which in a modern notation is denoted by the symbol G_1 when the original group G is transitive (and in what follows it is assumed that such is the case), was first studied in detail by A. L. Cauchy about 1845. Many of his results now seem trivial, but some of them are significant. In particular, he first demonstrated † the fundamental theorem that the order of G_1 equals the order of G divided by its degree. He showed ‡ further that when the G_1 of a transitive group is the direct product of transitive constituents the given group is imprimitive, a fact which is worth noting, since it is along a line which remained undeveloped until G. A. Miller recently treated § the subject of transitive groups whose subgroups consisting of the totality of operators which omit a particular letter are the direct product of transitive constituents. Cauchy also proved ‖ that if a transitive group G of degree n contains a cyclic substitution of order n, the number of such substitutions is a multiple of the order of G_1, a theorem of interest, since it is a special case of the more general one which states ¶ that in every group the number of letters in a cycle and its conjugates equals the order of the group. He announced erroneously that a simply transitive group is imprimitive unless all the constituents of G_1 are of the same degree.**

From 1845 until the recent work of G. A. Miller there was no direct attempt to study the properties of substitution groups by means of their G_1's,

* *Manuscrits d'Evariste Galois* (1908), page 39.

† *Oeuvres Complètes D'Augustin Cauchy*, First Series, Vol. 9, page 295.

‡ *Ibid.*, pages 317-320.

§ *Proceedings of the National Academy of Sciences*, Vol. 2, page 150.

‖ *Oeuvres Complètes d'Augustin Cauchy*, First Series, Vol. 9, page 469.

¶ *American Journal of Mathematics*, Vol. 44, page 122.

** G. A. Miller, *Bibliotheca Mathematica*, Series 3, Vol. 10, page 321.

although some theorems relating thereto appeared as by-products of other investigations. E. Mathieu showed * that for every value of p^m where p is a prime there is a doubly transitive group of degree p^m and order $p^m(p^m-1)$ contained in a triply transitive group of degree p^m+1 and order $(p^m+1)p^m(p^m-1)$. He demonstrated † likewise the existence of the two well-known five-fold transitive groups on 12 and 24 letters respectively. C. Jordon proved ‡ that in a simply transtive primitive group each prime that divides the order of one constituent of G_1 will divide the order of every such constituent. F. Rudio demonstrated § the fact that if G_1 omits more than one letter G is imprimitive, a theorem which he said had been communicated to him by Frobenius. H. W. Kuhn showed ‖ that the number of substitutions on the letters of any transitive group which are commutative with each of its substitutions equals the number of letters that are left unchanged by G_1. Shortly afterward it was proved ¶ that if the number of letters left unchanged by G_1 is α, then G_1 is invariant under an intransitive subgroup of G of order αg_1 (g_1 being the order of G_1), which has a regular transitive constituent on the letters omitted by G_1.

G. A. Miller has considered ** those groups whose G_1's are the direct products of transitive constituents showing, among other things, that in such a case if G_1 has a non-regular transitive constituent of lowest degree, there can be only one constituent of lowest degree; that if there is more than one constituent of lowest degree, these constituents are simply isomorphic regular groups, the degree of each of them being equal to the number of letters omitted by G_1; that if G_1 contains constituents of more than one degree, every constituent, except possibly those of lowest degree, is non-regular and the degree of every such constituent equals the sum of the degrees of all the constituents of lower degree increased by the number of letters omitted by G_1.

The same author treated *** also groups whose G_1's are of prime order or transitive, showing that the regular group or odd prime order p is always the G_1 of one, and only one, transitive group of degree $2p$; that in the special case when p is of the form $2^\alpha-1$ it is also the G_1 of one, and only one, group of degree $p+1$, and that it can not be the G_1 of any other transitive group;

---

* *Liouville's Journal*, Second Series, Vol. 5, pages 9-42.

† *Ibid.*, Vol. 6, pages 241-323.

‡ *Traité des Substitutions*, page 284.

§ *Crelle's Journal*, Vol. 102, page 3.

‖ *American Journal of Mathematics*, Vol. 23, page 173.

¶ G. A. Miller, *Bulletin of the American Mathematical Society*, Vol. 14, page 19.

** *Proceedings of the National Academy of Sciences*, Vol. 2, pages 150-152.

*** *Transactions of the American Mathematical Society*, Vol. 27, pages 137-145.

that the regular group of order 2 is the G_1 of the octic group and of the symmetric group of degree 3 and of no other group; that the number of transitive groups of degree $2k+1$ which have for their G_1's the group of order 2 and degree $2k$ is the same as the number of abstract abelian groups of order $2k+1$, k being an arbitrary positive integer; that every two different subgroups which are separately composed of all the substitutions which omit one letter of a transitive group of degree n generate a group of degree n; that a regular group of order 2^α, $\alpha > 2$, can not appear as the G_1 of a transitive group of degree $2^\alpha + 2$; that if a regular abelian group of order M is the G_1 of a transitive group of degree $m + h$, then m/h equals 1 or 2.

It is the object of this paper to consider further those transitive groups whose G_1's are regular, or more particularly the properties of those regular groups which may present themselves as the G_1's of transitive groups. Since the number of letters omitted by G_1 is a divisor of the degree of G as well as of the degree of G_1, it follows that if a regular group of degree m appears as the G_1 of a group G, then G is of degree $m + m/k$ where k is a divisor of m. We shall confine ourselves to those groups in which $m/k > 1$, that is to imprimitive groups. It is known * that every regular group of degree m may present itself as the G_1 of at least one transitive group of degree $2m$ and order $2m^2$; that is, if k is 1 there is no restriction on G_1. In § 2 of this article we shall determine those groups regular on m letters which may appear as the G_1's of groups of degree $m + m/2$; and in § 3 we shall consider some of the general properties of such regular groups where $k > 2$, proving in particular that there is no upper limit to the values which k may assume.

§ II. Regular Groups of Degree m which may appear as the G_1's of Groups of Degree $m + m/2$, $m/2 > 1$.

We know that G_1 is invariant under a subgroup of G which we shall call H_1. In the present case H_1 is of order $m^2/2$ and has two transitive constituents, one on the letters of G_1 which we shall denote by K_1, and a second which is regular on the $m/2$ letters of G which do not appear in G_1 and which we shall denote by C_1. In H_1, G_1 corresponds to the identity of C_1 but evidently to no other operator; and so K_1 is a subgroup of the holomorph of G_1, each defective † substitution omitting exactly $m/2$ letters. In other words the

---

* G. A. Miller, *Transactions of the American Mathematical Society*, Vol. 27, page 137.

† In speaking of substitution groups of degree m we shall use the term defective substitution to denote any substitution of degree α where $1 < \alpha < m$.

subgroup of K_1, which consists of all the substitutions of K_1 omitting a particular letter, and which we shall call G_{11} is a substitution group on $m/2$ letters; and since it is simply isomorphic with C_1 it is regular. Moreover, each coset of K_1 written with regard to G_1 contains exactly one substitution from G_{11} and one form is conjugate G_{11}' the substitutions having no letter in common. C_1 is therefore abelian, since from a coset of K_1 corresponding to any particular operator of C_1 we may choose a substitution on $m/2$ letters, while from the coset of K_1 corresponding to any other arbitrary operator of C_1 we may choose a substitution on the other $m/2$ letters of K_1, and hence their product in one order is identical with their product in the reverse order. Again, since G_{11} must appear as one of the constituents of a subgroup of index 2 in G_1 — this subgroup consisting of a simple isomorphism between G_{11} and G_{11}' — it follows that no regular substitution group of degree m can be the G_1 of a group of degree $m + m/2$ unless it involve an abelian subgroup of index 2, which is simply isomorphic with a subgroup of the group of isomorphisms I of this regular group of degree m. In particular, *no cyclic regular group of degree m except that of degree* 4 *can be the* G_1 *of a group of degree* $m + m/2$, $m/2 > 1$, since the order of the group of isomorphisms of a cyclic group of even order is always less than half the order of the cyclic group, unless the cyclic group have for its order 2^α, in which case the group of isomorphisms although of half the order of the cyclic group, is always non-cyclic whenever $\alpha > 2$. It is known that the cyclic regular group of degree 4 appears as the G_1 of exactly one group of degree 6.

Let us consider what abelian groups, outside the cyclic group mentioned, can appear as the G_1 of a transitive group of degree $m + m/2$. We have seen that G_1 contains a subgroup of index 2 consisting of a simple isomorphism between G_{11} and G_{11}'. In each coset of K_1 there appears some subsitution of G_{11}, say s and also a substitution of G_{11}', the latter substitution being that constituent of the substitution of G_1 whose other constituent is the inverse of s. Furthermore, these two defective substitutions in each coset of K_1 are conjugate under G_1; and since G_1 is abelian, it follows that every cycle in a substitution of G_1 is conjugate with every other cycle in that substitution. Hence s must be of order 2; otherwise we could pick out a cycle of G_1 such that the number of letters in it and its conjugates would exceed the order of G_1, which is impossible. Hence G_1 is of order 2^α and either of type $(1, 1, 1, \cdots)$ or of type $(2, 1, 1, \cdots)$. This condition is sufficient as well as necessary. For suppose we have a regular group of order 2^α and type $(1, 1, 1, \cdots)$ which we shall denote by G_1, although as yet we have not shown that it may actually possess the properties of the groups heretofore denoted by that symbol.

The group of isomorphisms of G_1 involves a subgroup of order $2^{\alpha-1}$ which leaves fixed each operator of a subgroup L of index 2 in G_1, while it multiplies successively an operator t of G_1 which is not in L by all the operators of L. This subgroup of I, therefore, when considered as a subgroup of the holomorph of G_1, omits the $m/2$ letters of one constituent of L and so is exactly the G_{11} referred to in the previous discussion. Hence G_1 may have associated with it a group similar to H_1 above. If we now extend this H_1 by an operator which together with the group of order $2^{\alpha-1}$ consisting of a simple isomorphism between G_{11} and C_1 will generate a group similar to G_1, we get a transitive group of degree $m + m/2$, under which the three subgroups C_1, G_{11} and G_{11}' are permuted according to the symmetric group of degree 3, H_1 corresponding to a subgroup of order 2. Hence the transitive group thus generated is of order $3m^2/2$. So we have a transitive group G of order $3m^2/2$ and degree $3m/2$. Its subgroup consisting of all the substitutions that omit a particular letter will be of order m and is therefore G_1. Moreover, from the method of construction we see that all such groups formed with a given G_1 are conjugate. The same reasoning would apply throughout if G_1 had been assumed to be of type $(2, 1, 1, \cdots)$. Hence we have the

THEOREM: *The necessary and sufficient condition that a regular abelian group of degree m appear as the G_1 of a group of degree $m + m/2$, $m/2 > 1$, is that it be of order 2^α and either of type $(1, 1, 1, \cdots)$ or of type $(2, 1, 1, \cdots)$; and every such group appears as the G_1 of exactly one group.*

It was noted above that, as a consequence of the fact that 2^α has primitive root only if $\alpha < 3$, we could say at once that the only regular cyclic group of degree m that could appear as the G_1 of a group of degree $m + m/2$, $m/2 > 1$ is the cyclic group of order 4. We can now show, conversely, that a number of the form 2^α can have primitive roots only if $\alpha = 1$ or 2. Otherwise, the I of a regular cyclic group G_1 of order 2^α, $(\alpha > 2)$ would be cyclic. Since the operator of order 2 in G_1 is characteristic, $2^{\alpha-2}$ of its transpositions would have to be permuted cyclically among themselves by a substitution of order $2^{\alpha-1}$ in the I of G_1. Since the subgroup L of index 2 in G_1 is characteristic, these $2^{\alpha-2}$ transpositions in G_1 are on the letters of one of the two regular constituents of L. Moreover, this substitution of order $2^{\alpha-1}$ in I can not involve any letters from the other constituent of L; for if it did the average number of letters in the holomorph of G_1 would be greater than $n - 1$. So I would be simply one of the constituents of L and by an argument similar to that employed above such a G_1 could appear as the subgroup consisting of the

totality of operators that omit a particular letter of a group of degree $m + m/2$, which is impossible. Hence the theorem.*

Let us now consider the non-abelian groups which will satisfy the condition under discussion. As before the two defective operators of each coset of K_1 are conjugate under G_1; and if s represents an operator of G_{11} which appears in a particular coset of K_1, then the other defective operator of this coset appears in G_1 in a substitution whose other constituent is s^{-1}. Hence every operator of G_1 outside its abelian subgroup of index 2 transforms each operator of this subgroup into its inverse. So G_1 is either a generalized dihedral group or a generalized dicyclic group. Moreover, we can show, just as before, that this condition is sufficient as well as necessary. Indeed this same argument has already been used to determine the groups of isomorphisms of the generalized dihedral and the generalized dicyclic groups.† Hence we have the

THEOREM: *The necessary and sufficient condition that a non-abelian regular group of degree m appear as the G_1 of group of degree $m + m/2$, $m/2 > 1$ is that it be either a generalized dihedral group or a generalized dicyclic group; and every such group appears as the G_1 of exactly one group.*

§ III. REGULAR GROUPS OF DEGREE m WHICH MAY APPEAR AS THE G_1'S OF GROUPS OF DEGREE $m + m/k$, $k > 2$, $m/k > 1$.

It is known that no abelian group of degree m can be the G_1 of a group of degree $m + m/k$, $k > 2$, $m/k > 1$. That such a non-abelian group may exist is evident from the group of isomorphisms of the non-cyclic group of order 9 when the former group is written on eight letters corresponding to the operators of order 3 in the latter group. There results a group of order 48 whose G_1 is the regular, non-cyclic group of order 6.‡

Considering further the case where k is 3 and retaining the notation of the last section, we notice that such a G_1 is invariant under a subgroup in G of order $m^2/3$, this subgroup H_1 arising from an m to 1 isomorphism between two transitive constituents: C_1 which is regular on the $m/3$ letters omitted by G_1, and K_1 which is a subgroup of the holomorph of G_1. Each defective substitution of K_1 omits exactly $m/3$ letters. If we denote by G_{11}

---

* C. F. Gauss, *Disquisitiones Arithmeticae*, § 89.

† G. A. Miller, *Philosophical Magazine*, Vol. 231, page 223.

‡ *Transactions of the American Mathematical Society*, Vol. 27, p. 145.

the subgroup of K_1 consisting of all the operators of K_1 which leave invariant a particular letter we see that G_{11} consists of a simple isomorphism between two regular constituents, each of order $m/3$; for it is simply isomorphic with the regular group C_1 of order $m/3$ and each of its operators, other than the identity, involves $2m/3$ letters; we see further that it is invariant under a subgroup of K_1 of order $m^2/9$, this latter group having a regular constituent C_{11} of order $m/3$ on the letters of K_1 omitted by G_{11}. There is, therefore, in G_1 a subgroup of index 3 which we denote by L. It is simply isomorphic with a subgroup of the group of isomorphism of G_1 and has 3 simply isomorphic regular constituents, one of them being C_{11}. Furthermore, as the argument is perfectly general, we have the

THEOREM: *A regular group of degree m can not appear as the G_1 of a group of degree $m + m/k$, $m/k > 1$, $k > 1$ unless it contains a subgroup of index k which is simply isomorphic with a subgroup of the group of isomorphisms of this regular group.*

Returning to the special case where k is 3, let us denote by L' that part of L which remains after dropping the constituent C_{11}. Now L' is a simple isomorphism between two regular constituents which we shall denote by L_1, and L_2. It has been shown that G_{11} is of order $m/3$ and each of its substitutions is commutative with every substitution of L'. Consequently G_{11} is either a one-to-one isomorphism between the conjoint of L_1 and the conjoint of L_2 or it is such that half of its substitutions interchange the letters of L_1 and L_2. We proceed to consider the second of these alternatives.

Since C_1 and C_{11} are simply isomorphic groups, so likewise are L' and G_{11}, these latter having no common operator except the identity and every operator of one being commutative with every operator of the other. Therefore $\{G_{11}, L'\}$ is of order $m^2/9$. Those operators of G_{11} which do not interchange L_1 and L_2 form an invariant subgroup of G_{11} of index 2. This subgroup, which we shall denote by M, consists of a simple isomorphism between a subgroup of index 2 of the conjoint of L_1 and a subgroup of index 2 of the conjoint of L_2. If we write $\{G_{11}, L'\}$ with regard to the invariant subgroup L', the multiplying operator in each coset being an operator of G_{11}, we see that each coset corresponding to an operator of M (except the coset L' itself) contains exactly two operators of degree $m/3$, one operator in each coset being on the letters of one of the conjugates of C_{11} under K_1 and the other operator in each coset being on the letters of the other conjugate of C_{11} under K_1. If one of the operators from M should be such that those of its constituents which are on the letters of L_1 actually constitute a substitution of L_1, then

every operator of M is such that all its cycles on the letters of L_1 actually constitute a substitution of L_1. We proceed to prove that this is impossible. If it were true, half the substitutions of L_1 would be in its central and consequently L_1, and therefore C_1, would be abelian. So every coset of K_1 written with regard to G_1 would go into itself when transformed by the operators of K_1. Suppose we consider a coset corresponding to one of the operators of M and consider in it a substitution of degree $2m/3$ which would omit the letters of L_1. Let us transform this substitution by one of the substitutions of G_{11} which interchanges the letters of L_1 and L_2. As this latter substitution omits the letters of C_{11}, it transforms the former substitution into itself; for the substitution in this coset on the letters of C_{11} and L_1 could not have its constituents on the letters of C_{11} the same as those constituents on the letters of C_{11} in the other substitution. This, however, is impossible, since the former substitution omits the letters of L_1 and is transformed into one omitting the letters of L_2. So we see that no substitution of M can be such that the totality of its cycles on the letters of L_1 will constitute a substitution of L_1. The same remarks apply to the letters of L_2. In other words, L_1 must have a subgroup of index 2 corresponding to a similar subgroup in its conjoint, the two subgroups having nothing in common except the identity. Furthermore, this shows that no substitution of the subgroup of index 2 of L_1 is invariant under L_1. For the subgroup of its conjoint corresponding to its subgroup of index 2 contains no substitution of L_1; but these are all the substitutions of the conjoint which do not interchange the two systems of letters of the subgroup of index 2 in L_1.

Returning to the subgroup L of G_1, we see that it contains a subgroup of index 2 which has 6 simply isomorphic regular constituents which we shall denote by $A, B, C, D, E,$ and F, where A and B are subgroups of L_1, C and D subgroups of L_2, and E and F subgroups of C_{11}. The group $\{G_{11}, L'\}$ is of degree $2m/3$ and order $m^2/9$; moreover its subgroup G_{111} consisting of all the substitutions which omit a particular letter is, as we saw, of degree $m/3$ and order $m/6$; it contains 2 simply isomorphic regular constituents. The substitutions of G_{111} can not interchange the letters of L_1 and L_2 since they do not involve all those letters. Hence one of the constituents of G_{111} is on letters of L_1 and the other on letters of L_2. Moreover, that one which is on the letters of L_1 involves half the letters of L_1; and since the letters of A and B are systems of imprimitivity of L_1, this constituent must be on the letters of one of them—say A, and similarly the other constituent must be on the letters of C or D—say C. Again, they are actually the conjoints of A and C

respectively, for they must be commutative with each substitution of $\{A \sim B \sim C \sim D\}$.* Moreover, the inverse of each substitution must be among the constituents of $\{A \sim B \sim C \sim D\}$ and consequently they are the groups A and C. That is, A, B, C, D, E, F are abelian groups. Likewise the 2 transitive constituents of the conjugate of G_{111} under $\{G_{11}, L'\}$ are B and D. We see, therefore, that M is a simple isomorphism between the four groups A, B, C, D,—different, of course, from that which appears in L. The subgroup $\{A \sim B \sim C \sim D \sim E \sim F\}$ is invariant under G_1 and these 6 subgroups give us systems of imprimitivity which are permuted under G_1 according to the regular non-cyclic group of order 6. The subgroup L permutes them thus: $AB \cdot CD \cdot EF$. The subgroup of G_1 containing a conjugate of C_{11} under K_1 will permute them according to a similar substitution of order 2; and likewise the subgroup containing the other conjugate of C_{11} under K_1. Moreover every coset of K_1 for which the multiplying operator is from M will permute them in a similar manner, while each of the remaining cosets of K_1 will permute them according to the other 6 operators in the subgroup of order 12 in the holomorph of the non-cyclic group of order 6. We know that each of the subgroups A, B, C, D, E, F, is abelian and that L_1 contains invariantly the abelian subgroup $\{A \sim B\}$, no operator of which except the identity is invariant under L_1. The group L_1 is, therefore, a generalized dihedral group and A is of odd order.†

Hence, considering G_1 as isomorphic with the non-cyclic regular group of order 6 according to which A, B, C, D, E, and F are permuted, we see that every operator in the cosets corresponding to these operators of order 2 is of order 2. Furthermore, these operators of order 2 interchange the 2 cosets corresponding to the 2 operators of order 3 and so none of them is commutative with any operator in the subgroup of G_1 of index 2 corresponding to the subgroup of order 3 in the group of order 6. Hence we see that G_1 is a generalized dihedral group. Moreover, the subgroup of index 2 of G_1 is of order 3^α and G_{11} is a subgroup of its group of isomorphisms of order $2 \cdot 3^{\alpha-1}$. Hence G_1 must be a generalized dihedral group whose subgroup of index 2 is of order 3^α and type $(1, 1, 1 \cdots)$.

That such a group can exist for all values of α is evident from the following considerations: represent the group of isomorphisms of an abelian group

---

* In general, if we are given r simply isomorphic groups, $\Gamma_1, \Gamma_2, \Gamma_3, \ldots, \Gamma_r$, we shall denote the groups consisting of a simple isomorphism between them by the symbol $\{\Gamma_1 \sim \Gamma_2 \sim \Gamma_3 \sim \ldots \sim \Gamma_r\}$.

† Burnside, *Theory of Groups* (1911), page 90.

of order 3^a and type $(1, 1, 1, \cdots)$ transitively on $3^a - 1$ letters. Its subgroup consisting of the totality of operators which omit a particular letter will be transitive, of degree $3^a - 3$ and order $(3^a - 3) \cdots (3^a - 3^{a-1})$; and its subgroup consisting of all the operators which omit a particular letter will in turn be of degree $(3^a - 9)$ and order $(3^a - 9) \cdots (3^a - 3^{a-1})$. Proceeding in this way we arrive at a regular group of degree $3^a - 3^{a-1}$ which is the G_1 of a transitive group of degree $3^a - 3^{a-2}$.

We have so far been considering the case where half the substitutions of G_{11} interchange the letters of L_1 and L_2. We shall show in what follows that in the other case, that is, where G_{11} consists of a simple isomorphism between the conjoint of L_1 and the conjoint of L_2, G_1 can have no subgroup of index 2. Hence we have the

THEOREM: *The necessary and sufficient condition that a regular group of degree m which contains a subgroup of index 2 shall appear as the G_1 of a group of degree $m + m/3$, $m/3 > 1$ is that it shall be a generalized dihegral group of order $2 \cdot 3^a$ whose abelian subgroup of index 2 is of type $(1, 1, 1, \cdots\cdots)$.*

As the existence proof employed above is perfectly general we have at once the additional

THEOREM: *For every prime value of k there exists an infinitude of regular groups of degree m which may appear as the G_1's of groups of degree $m + m/k$, $m/k > 1$.*

For composite values of k the matter is not so simple. Indeed, it is not certain that there is any composite value of k and a corresponding value of m for which a regular group of degree m may appear as the G_1 of a group of degree $m + m/k$, $m/k > 1$.

We have now to consider for $k = 3$ the possibilities when G_{11} consists of a simple isomorphism between the conjoints of L_1 and L_2. Since G_{11} omits the letters of C_{11}, each substitution of G_{11} is commutative with each substitution of L'; and since in the present case the substitutions of G_{11} do not interchange the two systems of L', it follows that G_{11} is an isomorphism between L_1 and L_2,—different, of course, from that which they enjoy in L'; consequently L_1, L_2, L, G_{11}, C_{11}, C_1 are all simply isomorphic abelian groups and G_1 contains an abelian subgroup of index 3 which is simply isomorphic with a subgroup of the group of isomorphisms of G_1. Since the letters of

L_1, L_2, C_{11} form systems of imprimitivity of G_1, L is an invariant subgroup of G_1 and its three systems are permuted by G_1 according to the group of order 3. Furthermore, since G_{11} is a subgroup of the group of isomorphisms of G_1 which leaves invariant each operator of L and transforms into itself each of the cosets of G_1 with regard to L—for it leaves the three systems of imprimitivity invariant—all the operators of G_1 outside of L are of the same order. So the number of Sylow subgroups of G_1 each having for its order a power of 3 is k, the order of G_1 being $3^\alpha k$, k prime to 3. If G_1 contained a subgroup of index 2, the number of its Sylow subgroups each having for its order a power of 3 would be $k/2$. As this is impossible, such a G_1 could not have a subgroup of index 2.

Since every operator of G_1 outside of L has for its order a power of 3, L can have no operator invariant under G_1 unless that operator be of order 3^α ($\alpha \geqq 0$). Hence we have the

THEOREM: *A regular group of degree m containing no subgroup of index 2 can appear as the G_1 of a group of degree $m + m/3$ only if this regular group contains an invariant abelian subgroup of index 3 which is simply isomorphic with a subgroup in the group of isomorphisms of the given group of degree m and which contains no operator invariant under the given regular group except the identity and, possibly, operators whose orders are powers of 3.*

As examples of such regular groups we have the non-abelian group of order 21, the non-abelian group of order 39, the tetrahedral group, the group of order 48 which contains an abelian subgroup of order 16 and type (1,1,1,1) and permutes these 15 substitutions of order 2 according to 5 cycles of 3 each, the group of order 36 which contains an invariant abelian subgroup of order 12 whose invariants are 6 and 2 and whose three operators of order 2 are permuted cyclically by the entire group, and the non-abelian group of order 27 containing no operator of order 9. The first two groups mentioned are examples of the

THEOREM: *Every non-abelian regular group of order $3p$ where p is any odd prime appears as the G_1 of a transitive group of degree $4p$.*

To prove this we have only to note that the group of isomorphisms of every such group G_1 contains an operator of order p which leaves invariant every substitution of the given group of order $3p$ except those of order 3 and permutes these latter in 2 cycles each of order p. Hence the holomorph of

G_1 contains a subgroup of order $3p^2$ whose subgroup consisting of the totality of substitutions which leave invariant a particular letter is of order p, consisting of a simple isomorphisms between 2 of the constituents of the subgroup of order p of G_1. This subgroup may serve as the K_1 mentioned above. It has three systems of imprimitivity which we may denote by A, B, and C, each involving p letters and it permutes them according to the cyclic group on three letters. We may construct a $3p$ to 1 isomorphism between K_1 and a regular group on p additional letters thereby forming H_1 which will have a subgroup of order p involving 3 constituents, A, B, and one on the p added letters which we may denote by D. We may extend this group by means of a substitution of order 3 which gives a non-abelian group G_1' conjugate with G_1. The three systems of imprimitivity A, B, and D of G_1' are permuted by it according to the cyclic group of degree 3. Hence G_1 and G'_1 generate a transitive group of degree $4p$ which permutes the four systems A, B, C, and D according to the alternating group and its subgroup consisting of all its operators that leave invariant a particular letter is the given G_1.

UNIVERSITY OF ILLINOIS.
URBANA, ILLINOIS.

Generalized Green's Functions for Compatible Differential Systems.

By W. W. Elliott.

1. *Introduction.* If the fundamental homogeneous differential system *

$$
\begin{aligned}
&L(u) = l_n(x)u^{(n)}(x) + \cdots + l_1(x)u'(x) + l_0(x)u(x) = 0;\\
(\mathrm{I}) \quad &U_1(u) = \alpha_{11}u(a) + \alpha_{12}u'(a) + \cdots + \alpha_{1n}u^{(n-1)}(a) + \beta_{11}u(b) + \\
&\qquad\qquad \cdots + \beta_{1n}u^{(n-1)}(b) = 0,\\
&\qquad\qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&U_n(u) = \alpha_{n1}u(a) + \alpha_{n2}u'(a) + \cdots + \alpha_{nn}u^{(n-1)}(a) + \beta_{n1}u(b) + \\
&\qquad\qquad \cdots + \beta_{nn}u^{(n-1)}(b) = 0,
\end{aligned}
$$

is incompatible, its adjoint † system

$$
\begin{aligned}
&M(v) = (-1)^n \frac{d^n}{dx^n}[l_n(x)v(x)] + \cdots - \frac{d}{dx}[l_1(x)v(x)] + l_0(x)v(x)\\
&\quad = 0; \quad V_1(v) = \gamma_{11}v(a) + \cdots + \gamma_{1n}v^{(n-1)}(a) + \delta_{11}v(b) + \cdots\\
(\mathrm{II}) \qquad &\qquad\qquad + \delta_{1n}v^{(n-1)}(b),\\
&\qquad\qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\qquad V_n(v) = \gamma_{n1}v(a) + \cdots + \gamma_{nn}v^{(n-1)}(a) + \delta_{n1}v(b) + \cdots\\
&\qquad\qquad + \delta_{nn}v^{(n-1)}(b),
\end{aligned}
$$

is also incompatible,‡ and they both possess Green's functions in the ordinary sense.§

The Green's function, $G(xy)$, for system (I) is defined by the following conditions,

(III) (a) $\qquad G(xy), \partial G(xy)/\partial x, \cdots, \partial^{n-1}G(xy)/\partial x^{n-1},$

exist and are continuous for $a \leq x \leq b$, $a \leq y \leq b$,

(b) $\qquad \partial^{n-1}G(xy)/\partial x^{n-1}, \ \partial^n G(xy)/\partial x^n,$

---

* M. Bôcher, *Annals of Mathematics*, ser. 2, Vol. 13 (1911), p. 71.

† M. Bôcher, *Les Méthodes de Sturm*, p. 34.

‡ M. Bôcher, *Les Méthodes de Sturm*, p. 34.

§ M. Bôcher, *Annals of Mathematics*, ser. 2, Vol. 13 (1911), p. 71.

exist and are continuous for $a \leq x < y$, $y < x \leq b$,

(c) $\partial^{n-1}G(y+, y)/\partial x^{n-1} - \partial^{n-1}G(y-, y)/\partial x^{n-1} = 1/l_n(y)$,

(d) $L_x[G(xy)] = 0, \quad a \leq x < y, \; y < x \leq b$,

(e) $U_p[G(xy)] = 0, \quad [p = 1, 2, \cdots, n]$.

These conditions determine a unique function, which is given by the explicit formula,*

$$\text{(IV)} \quad G(xy) = \frac{\begin{vmatrix} H(xy) & u_1(x) & \cdots & u_n(x) \\ U_1(H) & U_1(u_1) & \cdots & U_1(u_n) \\ \cdots & \cdots & \cdots & \cdots \\ U_n(H) & U_n(u_1) & \cdots & U_n(u_n) \end{vmatrix}}{\begin{vmatrix} U_1(u_1) & \cdots & U_1(u_n) \\ \cdots & \cdots & \cdots \\ U_n(u_1) & \cdots & U_n(u_n) \end{vmatrix}},$$

in which

$$\text{(V)} \quad H(xy) = \frac{\operatorname{sgn}(x-y)}{2l_n(y)} \cdot \frac{\begin{vmatrix} u_1(y) & \cdots & u_n(y) \\ u_1'(y) & \cdots & u_n'(y) \\ \cdots & \cdots & \cdots \\ u_1^{(n-2)}(y) & \cdots & u_n^{(n-2)}(y) \\ u_1(x) & \cdots & u_n(x) \end{vmatrix}}{\begin{vmatrix} u_1(y) & \cdots & u_n(y) \\ u_1'(y) & \cdots & u_n'(y) \\ \cdots & \cdots & \cdots \\ u_1^{(n-1)}(y) & \cdots & u_n^{(n-1)}(y) \end{vmatrix}},$$

where

$$\operatorname{sgn} x = (1, x < 0; \; 0, x = 0; \; -1, x > 0).$$

Green's functions of this general type have been studied by Burkhardt, Bôcher, Birkhoff, Bounitzky, and many others.†

Among the properties of the function $G(xy)$ the following are perhaps the most fundamental,

---

* For formulas like these, except for change in notation, see Westfall's dissertation, Göttingen (1905).

† G. D. Birkhoff, *Transactions of the American Mathematical Society*, Vol. 9 (1908), p. 373; M. E. Bounitzky, *Journal de Mathématiques* (1909), p. 65; M. Bôcher, *Bulletin of the American Mathematical Society*, Vol. 7 (1901), p. 297; *Annals of Mathematics*, ser. 2, Vol. 13 (1911), p. 71; M. H. Burkhardt, *Bulletin de la Société Mathématique de France*, Vol. 22 (1894), p. 71.

(A) The solution of the semi-homogeneous system

$$L(u) = f(x), \quad U_p(u) = 0 \qquad (p = 1, \cdots, n), \tag{VI}$$

$f(x)$ being any continuous function, is given by *

$$u(x) = \int_a^b G(xs)\, f(s)\, ds.$$

Also, conversely, if for every continuous $f(x)$ the solution of (VI) can be written in the form

$$u(x) = \int_a^b G_1(xs) f(s)\, ds,$$

then $$G_1(xs) = G(xs).$$

(B) If $H(xy)$ denotes the Green's function for System (II), then

$$H(xy) = G(yx).\dagger$$

The existence of generalized Green's functions for certain special types of compatible differential systems has been previously shown.‡

It is the purpose of this paper to demonstrate the existence and characteristic properties of generalized Green's functions for general compatible differtial systems of type I.

2. *Existence of Generalized Green's Functions.* The fundamental homogeneous differential system considered will be of the form,§

$$\begin{aligned}
&L(u) = l_n(x)u^{(n)}(x) + \cdots + l_1(x)u'(x) + l_0(x)u(x) = 0;\\
&U_1(u) = \alpha_{11}u(a) + \alpha_{12}u'(a) + \cdots + \alpha_{1n}u^{(n-1)}(a) + \beta_{11}u(b)\\
&\qquad\qquad + \beta_{12}u'(b) + \cdots + \beta_{1n}u^{(n-1)}(b) = 0,\\
&\quad\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&U_n(u) = \alpha_{n1}u(a) + \alpha_{n2}u'(a) + \cdots + \alpha_{nn}u^{(n-1)}(a) + \beta_{n1}u(b)\\
&\qquad\qquad + \beta_{n2}u'(b) + \cdots + \beta_{nn}u^{(n-1)}(b) = 0,
\end{aligned}$$

where it is assumed $l_n(x), \cdots, l_0(x)$ are continuous functions, real or complex, of the real variable x in the interval $a \leq x \leq b$, $l_n(x)$ does not vanish

---

* M. Bôcher, *Les Méthodes de Sturm*, p. 103.

† M. Bôcher, *Les Méthodes de Sturm*, p. 104.

‡ D. Hilbert, *Göttinger Nachrichten* (1904), p. 213; F. Klein, *Mathematische Annalen*, Vol. 21 (1882), p. 141; W. Westfall, *Annals of Mathematics*, ser. 2, Vol. 10 (1909), p. 177; M. E. Bounitzky, *Journal de Mathématiques* (1909), p. 65; A. Myller, Inaugural-Dissertation, Göttingen (1906).

§ G. D. Birkhoff, *Transactions of the American Mathematical Society*, Vol. 9 (1908), p. 373.

in this interval, each $l_k(x)$ has a continuous derivative of order k, and the n sets of $2n$ constants, denoted by the α's and β's, are linearly independent.

The above system will be denoted briefly by

$$L(u) = 0, \quad U_p(u) = 0 \qquad (p = 1, \cdots, n). \tag{1}$$

The system adjoint to (1) will be denoted by

$$M(v) = 0, \quad V_p(v) = 0 \qquad (p = 1, \cdots, n). \tag{2}$$

System (1) is called self-adjoint * if $L(u) = M(u)$, and if the sets of expressions

$$[V_1(u), \cdots, V_n(u)] \quad \text{and} \quad [U_1(u), \cdots, U_n(u)]$$

are linearly dependent.

System (1) is called Hermitian if

$$\bar{L}(u) = M(u),$$

and if the sets of expressions

$$[V_1(u), \cdots, V_n(u)] \quad \text{and} \quad [\bar{U}_1(u), \cdots, \bar{U}_n(u)]$$

are linearly dependent, the bars denoting conjugates of the coefficients only.

A Hermitian differential system obviously reduces to a self-adjoint system if all the coefficients are real. All results for Hermitian differential systems will then include those for self-adjoint systems if the coefficients are real.

We shall assume that system (1) is compatible of order k. It follows † that system (2) is compatible of order k. A complete set of k linearly independent solutions of (1) and (2) will be denoted respectively by

$$u_1(x), \cdots, u_k(x) \quad \text{and} \quad v_1(x), \cdots, v_k(x).$$

The general solution of (1) may be written in the form

$$u(x) = \sum_{i=1}^{k} A_i u_i(x),$$

where the A_i's are arbitrary constants.

On the assumption that (1) is compatible, the semi-homogeneous system

$$L(u) = f(x), \quad U_p(u) = 0 \qquad (p = 1, \cdots, n), \tag{3}$$

---

* M. Bôcher, *Les Méthodes de Sturm*, p. 39.

† M. Bôcher, *Les Méthodes de Sturm*, p. 34.

$f(x)$ being any continuous function, will have a solution with a continuous nth derivative if and only if *

$$\int_a^b v_p(x)f(x)\,dx = 0 \qquad (p = 1, \cdots, k). \tag{4}$$

If (4) is satisfied, from the analogy of the ordinary † Green's function we are led to seek the existence of a continuous function $G(xs)$ such that every solution of (3) may be written in the form

$$u(x) = \sum_{i=1}^{k} A_i u_i(x) + \int_a^b G(xs)f(s)\,ds. \tag{5}$$

If there exists such a continuous $G(xs)$, then we shall call $G(xs)$ a *Generalized Green's Function* for the compatible differential system (1).

Let $K(xy)$ be a function satisfying the following conditions,‡

(6) (a) $\quad K(xy),\ \partial K(xy)/\partial x, \cdots, \partial^{n-2}K(xy)/\partial x^{n-2},$

exist and are continuous for $a \leq x \leq b,\ a \leq y \leq b,$

(b) $\quad \partial^{n-1}K(xy)/\partial x^{n-1},\ \partial^n K(xy)/\partial x^n,$

exist and are continuous for $a \leq x < y,\quad y < x \leq b,$

(c) $\quad \partial^{n-1}K(y+, y)/\partial x^{n-1} - \partial^{n-1}K(y-, y)/\partial x^{n-1} = 1/l_n(y),$

(d) $\quad L_x[K(xy)] = 0,\quad a \leq x < y,\quad y < x \leq b.$

Such functions exist and are given by the formula §

$$K(xy) = \frac{\operatorname{sgn}(x-y)}{2l_n(y)} \cdot \frac{\begin{vmatrix} u_1(y) & \cdots & u_n(y) \\ u_1'(y) & \cdots & u_n'(y) \\ \cdots & \cdots & \cdots \\ u_1^{n-2}(y) & \cdots & u_n^{n-2}(y) \\ u_1(x) & \cdots & u_n(x) \end{vmatrix}}{\begin{vmatrix} u_1(y) & \cdots & u_n(y) \\ u_1'(y) & \cdots & u_n'(y) \\ \cdots & \cdots & \cdots \\ u_1^{(n-1)}(y) & \cdots & u_n^{(n-1)}(y) \end{vmatrix}} \tag{7}$$

* M. Bôcher, *Les Méthodes de Sturm*, p. 36.

† See Introduction.

‡ These conditions are the same as III (a, b, c, d) in the Introduction.

§ G. D. Birkhoff, *Transactions of the American Mathematical Society*, Vol. 9 (1908), p. 373.

THEOREM I. *A generalized Green's function for the compatible differential system* (1) *exists and may be expressed by the formula* *

$$G(xs) = \frac{\begin{vmatrix} K(xs) & u_{k+1}(x) & \cdots & u_n(x) \\ U_1(K) & U_1(u_{k+1}) & \cdots & U_1(u_n) \\ \cdots & \cdots & \cdots & \cdots \\ U_{n-k}(K) & U_{n-k}(u_{k+1}) & \cdots & U_{n-k}(u_n) \end{vmatrix}}{\begin{vmatrix} U_1(u_{k+1}) & \cdots & U_1(u_n) \\ \cdots & \cdots & \cdots \\ U_{n-k}(u_{k+1}) & \cdots & U_{n-k}(u_n) \end{vmatrix}} .$$

By way of proof, let

$$u_0(x) = \int_a^b K(xs) f(s)\, ds;$$

then

$$L[u_0(x)] = f(x).$$

Every solution of $L(u) = f(x)$ is given by

$$u(x) = u_0(x) + \sum_{i=1}^{n} c_i u_i(x), \tag{8}$$

where the constants c_i are arbitrary. We seek to determine these constants so that $u(x)$ will satisfy the boundary conditions.

Since it is assumed that (1) is compatible of order k and (3) has a solution the rank † of each of the matrices

$$\left\| \begin{matrix} U_1(u_1) & \cdots & U_1(u_n) \\ \cdots & \cdots & \cdots \\ U_n(u_1) & \cdots & U_n(u_n) \end{matrix} \right\| , \tag{9}$$

and

$$\left\| \begin{matrix} U_1(u_1) & \cdots & U_1(u_n) & -U_1(u_0) \\ \cdots & \cdots & \cdots & \cdots \\ U_n(u_1) & \cdots & U_n(u_n) & -U_n(u_0) \end{matrix} \right\| , \tag{10}$$

is $(n-k)$.

Assume the solutions so arranged that $u_p(x)$ $[p = 1, \cdots, k]$, satisfy

---

* This does not imply that all generalized Green's functions for System (1) are given by this formula. In order to obtain this formula, certain arrangements of solutions and the boundary conditions are assumed. These are made clear in the proof.

† M. Bôcher, *Les Méthodes de Sturm*, p. 21.

the boundary conditions and that the $(n-k)$-rowed determinant in the upper right hand corner of (9) fails to vanish. Then the system of equations

$$\begin{aligned} c_{k+1}U_1(u_{k+1}) + \cdots + c_nU_1(u_n) &= -U_1(u_0), \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ c_{k+1}U_n(u_{k+1}) + \cdots + c_nU_n(u_n) &= -U_n(u_0), \end{aligned}$$

is obtained by applying the boundary conditions to (8) and assuming they are satisfied by $u(x)$. These equations may be solved for c_i's $(i = k+1, \cdots, n)$, and upon substituting the values thus found in (8) and simplifying, we get

$$u(x) = \sum_{i=1}^{k} c_iu_i(x) + \frac{\begin{vmatrix} \int_a^b K(xs)f(s)\,ds & u_{k+1}(x) & \cdots & u_n(x) \\ U_1\left[\int_a^b K(xs)f(s)\,ds\right] & U_1(u_{k+1}) & \cdots & U_1(u_n) \\ \cdots\cdots\cdots\cdots & & & \\ U_{n-k}\left[\int_a^b K(xs)f(s)\,ds\right] & U_{n-k}(u_{k+1}) & \cdots & U_{n-k}(u_n) \end{vmatrix}}{\begin{vmatrix} U_1(u_{k+1}) & \cdots & U_1(u_n) \\ \cdots\cdots\cdots & & \\ U_{n-k}(u_{k+1}) & \cdots & U_{n-k}(u_n) \end{vmatrix}}$$

Since the derivatives of the function $\int_a^b K(xs)f(s)\,ds$ up through the $(n-1)$th may be obtained correctly by formally differentiating under the integral sign, the equation above may be written

$$u(x) = \sum_{i=1}^{k} c_iu_i(x) + \int_a^b \frac{\begin{vmatrix} K(xs) & u_{k+1}(x) & \cdots & u_n(x) \\ U_1(K) & U_1(u_{k+1}) & \cdots & U_1(u_n) \\ \cdots\cdots\cdots\cdots & & & \\ U_{n-k}(K) & U_{n-k}(u_{k+1}) & \cdots & U_{n-k}(u_n) \end{vmatrix}}{\begin{vmatrix} U_1(u_{k+1}) & \cdots & U_1(u_n) \\ \cdots\cdots\cdots & & \\ U_{n-k}(u_{k+1}) & \cdots & U_{n-k}(u_n) \end{vmatrix}} f(s)\,ds,$$

which demonstrates that the function

$$(11) \qquad G(xs) = \frac{\begin{vmatrix} K(xs) & u_{k+1}(x) & \cdots & u_n(x) \\ U_1(K) & U_1(u_{k+1}) & \cdots & U_1(u_n) \\ \cdots\cdots\cdots\cdots & & & \\ U_{n-k}(K) & U_{n-k}(u_{k+1}) & \cdots & U_{n-k}(u_n) \end{vmatrix}}{\begin{vmatrix} U_1(u_{k+1}) & \cdots & U_1(u_n) \\ \cdots\cdots\cdots & & \\ U_{n-k}(u_{k+1}) & \cdots & U_{n-k}(u_n) \end{vmatrix}}$$

may be taken as a generalized Green's function for the compatible differential system (1).

Any function given by formula (11) will satisfy the properties of the ordinary Green's function given by III (a, b, c, d) in the introduction and

$$U_p[G(xs)] = 0 \qquad [p = 1, \cdots, (n-k)].$$

Also it is seen that the ordinary Green's function may be considered as a special case of $G(xs)$ when the order of compatibility of (1) is zero.

THEOREM II. *The generalized Green's function for the compatible system* (1) *is not unique. If* $G_1(xs)$ *is any one generalized Green's function, then every generalized Green's function* $G(xs)$ *is given by the formula,*

$$G(xs) = G_1(xs) + \sum_{i=1}^{k} \phi_i(x) v_i(s) + \sum_{i=1}^{k} u_i(x) \psi_i(s),$$

where the functions ϕ_i *and* ψ_i *are arbitrary continuous functions.*

Let

$$f(x) = \begin{vmatrix} g(x) & \bar{v}_1(x) \cdots \bar{v}_k(x) \\ \int_a^b v_1(s) g(s)\, ds & v_{1,1} \cdots v_{1,k} \\ \cdots\cdots\cdots & \cdots\cdots\cdots \\ \int_a^b v_k(s) g(s)\, ds & v_{k,1} \cdots v_{k,k} \end{vmatrix}, \tag{12}$$

where $g(x)$ is any continuous function and

$$v_{p,q} = \int_a^b v_p(s)\, \bar{v}_q(s)\, ds.$$

On multiplying (12) by $v_p(x)$ $(p = 1, \cdots, k)$, and integrating with respect to x from a to b, it is seen that $f(x)$ satisfies condition (4).

Let $G(xs)$ and $G_1(xs)$ be any generalized Green's functions * for system (1), and form the difference

$$D(xs) = G(xs) - G_1(xs). \tag{13}$$

If equation (13) is multiplied by $f(s)$, integrated and simplified by making use of equation (5) it is seen that

$$\int_a^b D(xs) f(s)\, ds = \sum_{i=1}^{k} B_i u_i(x), \tag{14}$$

the B_i's being arbitrary constants.

* These functions may or may not be distinct.

From (12) and (14) it follows that

$$\int_a^b \Phi(xs)g(s)\,ds = \sum_{i=1}^{k} B_i u_i(x), \tag{15}$$

where

$$\Phi(xs) = \begin{vmatrix} D(xs) & \int_a^b D(xs)\bar{v}_1(s)\,ds & \cdots & \int_a^b D(xs)\bar{v}_k(s)\,ds \\ v_1(s) & v_{1,1} & \cdots & v_{1,k} \\ \cdots & \cdots & \cdots & \cdots \\ v_k(s) & v_{k,1} & \cdots & v_{k,k} \end{vmatrix}. \tag{16}$$

On multiplying (15) by $\bar{u}_p(x)$ $(p = 1, \cdots, k)$, and integrating with respect to x from a to b we obtain the following equations

$$\begin{aligned} \int_a^b\int_a^b \bar{u}_1(x)\,\Phi(xs)g(s)\,ds\,dx &= \sum_{i=1}^{k} B_i \int_a^b u_i(x)\bar{u}_1(x)\,dx, \\ &\cdots\cdots\cdots\cdots \\ \int_a^b\int_a^b \bar{u}_k(x)\,\Phi(xs)g(s)\,ds\,dx &= \sum_{i=1}^{k} B_i \int_a^b u_i(x)\bar{u}_k(x)\,dx. \end{aligned} \tag{17}$$

The determinant of the coefficients of the B_i's in system (17) does not vanish since it is the gramian of k linearly independent functions. Solving these equations for the B_i's and substituting the values thus found in (15) we obtain, after simplification,

$$\int_a^b \begin{vmatrix} \Phi(xs) & u_1(x) & \cdots & u_k(x) \\ \int_a^b \bar{u}_1(t)\Phi(ts)\,dt & u_{1,1} & \cdots & u_{k,1} \\ \cdots & \cdots & \cdots & \cdots \\ \int_a^b \bar{u}_k(t)\Phi(ts)\,dt & u_{1,k} & \cdots & u_{k,k} \end{vmatrix} g(s)\,ds = 0,$$

where $g(s)$ is any arbitrary continuous function.

Then,

$$\begin{vmatrix} \Phi(xs) & u_1(x) & \cdots & u_k(x) \\ \int_a^b \bar{u}_1(t)\,\Phi(ts)\,dt & u_{1,1} & \cdots & u_{k,1} \\ \cdots & \cdots & \cdots & \cdots \\ \int_a^b \bar{u}_k(t)\,\Phi(ts)\,dt & u_{1,k} & \cdots & u_{k,k} \end{vmatrix} = 0. \tag{18}$$

The coefficient of $\Phi(xs)$ in (18) does not vanish. Hence (18) may be solved for Φ, giving *

$$\Phi(xs) = \sum_{i=1}^{k} u_i(x)\alpha_i(s). \tag{19}$$

* The expressions for the functions α_i $(i = 1, \ldots, k)$ may be obtained from the above solution.

Upon substituting the value of $\Phi(xs)$ given by (16) in (19) and solving for $D(xs)$, we get

$$D(xs) = \sum_{i=1}^{k} \phi_i(x)v_i(s) + \sum_{i=1}^{k} u_i(x)\psi_i(s), \tag{20}$$

which gives the general form of the function D.

The function $G(xs)$ must be then of the form

$$G(xs) = G_1(xs) + \sum_{i=1}^{k} \phi_i(x)v_i(s) + \sum_{i=1}^{k} u_i(x)\psi_i(s). \tag{21}$$

If (21) is multiplied by $f(s)$ and integrated with respect to s from a to b, it is seen that if $G_1(xs)$ is any one generalized Green's function, $G(xs)$ will also serve as a generalized Green's function when $\phi_i(x)$ and $\psi_i(s)$ $(i=1, \cdots, k)$, are any arbitrary continuous functions.

3. *Generalized Green's Functions with Property of Symmetry.* It is desirable to define a generalized Green's function which shall possess for adjoint systems the same property of symmetry as the ordinary Green's function.* With this end in view, we prove several lemmas.

LEMMA I. *If $K(xy)$ is any function satisfying conditions III (a, b, c, d) of the introduction, and $v(x)$ is any solution of the equation $M(v)=0$, then*

$$\sum_{p=1}^{2n} U_p[K(xy)]\, V_{2n+1-p}[v(x)] = v(y).$$

To prove this, apply Green's theorem,† letting $v = v(x)$ and $u = K(xy)$. We get

$$\int_a^b \{v(x)L_x[K(xy)] - K(xy)M[v(x)]\}\, dx = [P(K,v)]_a^{y-} + [P(K,v)]_{y+}^b$$

Since the integral vanishes,

$$[P(K,v)]_a^{y-} + [P(K,v)]_{y+}^b = 0.$$

Whence

$$P[(K,v)]_a = [P(K,v)]_{y-}^{y+} = l_n(y)v(y)\left[\frac{\partial^{n-1}}{\partial x^{n-1}} K(x,y)\right]_{y-}^{y+} = v(y).$$

---

* See Introduction.

† For further details of such a proof, see M. Bôcher, *Les Méthodes de Sturm*, p. 104.

But

$$[P(K,v)]_a^b = \sum_{p=1}^{2n} U_p\,[K(xy)]\; V_{2n+1-p}(v).$$

Therefore

$$\sum_{p=1}^{2n} U_p\,[K(xy)]\; V_{2n+1-p}(v) = v(y),$$

as we wished to show.

Define the function $N(xy)$ by the formula

$$N\begin{Bmatrix} v_1(y) & \cdots & v_k(y) \\ \psi_1(x) & \cdots & \psi_k(x) \end{Bmatrix} = \frac{\begin{vmatrix} 0 & \psi_1(x) & \cdots & \psi_k(x) \\ v_1(y) & \int_a^b v_1(s)\psi_1(s)\,ds & \cdots & \int_a^b v_1(s)\psi_k(s)\,ds \\ \cdots & \cdots & \cdots & \cdots \\ v_k(y) & \int_a^b v_k(s)\psi_1(s)\,ds & \cdots & \int_a^b v_k(s)\psi_k(s)\,ds \end{vmatrix}}{\begin{vmatrix} \int_a^b v_1(s)\psi_1(s)\,ds & \cdots & \int_a^b v_1(s)\psi_k(s)\,ds \\ \cdots & \cdots & \cdots \\ \int_a^b v_k(s)\psi_1(s)\,ds & \cdots & \int_a^b v_k(s)\psi_k(s)\,ds \end{vmatrix}},$$

where the ψ_i's are any functions whatever, which give the determinant in the denominator of the above fraction a value different from zero.*

It is readily seen that *the function N is independent of linear combinations of the linearly independent solutions $(v_1, \cdots, v_k)$, and also of any particular linear combination of the ψ_i's after this set of functions has been chosen.*

LEMMA II. *The system*

$$(22) \qquad L[K_1(xy)] = N(xy), \quad U_p[K_1(xy)] = -U_p(K), \quad (p = 1, \cdots, n),$$

has a solution in x which possesses a continuous nth derivative taken with respect to x.

Solutions of the above system with a continuous nth derivative exist if †

$$(23) \qquad \int_a^b v_i(x)\,N(xy)\,dx = -\sum_{p=1}^{n} U_p(K)\,V_{2n+1-p}(v_i) \qquad (i = 1, \cdots, k).$$

---

* If this condition is satisfied, the v_i's and ψ_i's may be transformed linearly into biorthogonal sets.

† M. Bôcher, *Les Méthodes de Sturm*, p. 36.

But

$$(24)\qquad \int_a^b v_i(x)N(xy)\,dx = \frac{\begin{vmatrix} 0 & \int_a^b v_i(s)\psi_1(s)\,ds & \cdots & \int_a^b v_i(s)\psi_k(s)\,ds \\ v_1(y) & \int_a^b v_1(s)\psi_1(s)\,ds & \cdots & \int_a^b v_1(s)\psi_k(s)\,ds \\ \cdot & \cdot & \cdots & \cdot \\ v_k(y) & \int_a^b v_k(s)\psi_1(s)\,ds & \cdots & \int_a^b v_k(s)\psi_k(s)\,ds \end{vmatrix}}{\begin{vmatrix} \int_a^b v_1(s)\psi_1(s)\,ds & \cdots & \int_a^b v_1(s)\psi_k(s)\,ds \\ \cdot & \cdots & \cdot \\ \int_a^b v_k(s)\psi_1(s)\,ds & \cdots & \int_a^b v_k(s)\psi_k(s)\,ds \end{vmatrix}}$$

$$= \frac{\begin{vmatrix} -v_i(y) & 0 & \cdots & 0 \\ v_1(y) & \int_a^b v_1(s)\psi_1(s)\,ds & \cdots & \int_a^b v_1(s)\psi_k(s)\,ds \\ \cdot & \cdot & \cdots & \cdot \\ v_k(y) & \int_a^b v_k(s)\psi_1(s)\,ds & \cdots & \int_a^b v_k(s)\psi_k(s)\,ds \end{vmatrix}}{\begin{vmatrix} \int_a^b v_1(s)\psi_1(s)\,ds & \cdots & \int_a^b v_1(s)\psi_k(s)\,ds \\ \cdot & \cdots & \cdot \\ \int_a^b v_k(s)\psi_1(s)\,ds & \cdots & \int_a^b v_k(s)\psi_k(s)\,ds \end{vmatrix}}$$

$$= -v_i(y) \qquad (i = 1, \cdots, k).$$

Since

$$\sum_{p=1}^{2n} U_p(K)\, V_{2n+1-p}(v_i) = \sum_{p=1}^{n} U_p(K)\, V_{2n+1-p}(v_i) \qquad (i = 1, \cdots, k),$$

from (24) and lemma (I) it is seen that condition (23) is satisfied and therefore system (22) has a solution.

The solution K_1 of (22) is not unique but is determined except for added solutions of the homogeneous system corresponding to (22).

Lemmas (I) and (II) hold for the adjoint systems when v_i is replaced by u_i and ψ_i by ϕ_i $(i = 1, \cdots, k)$, if the ϕ_i's are any functions * which give the determinant,

$$\begin{vmatrix} \int_a^b u_1(s)\phi_1(s)\,ds & \cdots & \int_a^b u_1(s)\phi_k(s)\,ds \\ \cdot & \cdots & \cdot \\ \int_a^b u_k(s)\phi_1(s)\,ds & \cdots & \int_a^b u_k(s)\phi_k(s)\,ds \end{vmatrix}$$

a value different from zero.

---

* The ϕ_i's and ψ_i's used in this section should not be confused with those used in Section (2).

Let

$$G(xy) = K(xy) + K_1(xy). \tag{25}$$

Among the solutions chosen for K_1 there is always one for which

$$\int_a^b \phi_i(x) G(xy)\, dx = 0 \qquad (i = 1, \cdots, k). \tag{26}$$

Hence the following theorem has been demonstrated:

THEOREM III. *If the system* (1) *is compatible, there exists a function satisfying the following conditions:*

(27) (a) $\quad G(xy),\ \partial G(xy)/\partial x,\ \cdots,\ \partial^{n-2} G(xy)/\partial x^{n-2}$

exist and are continuous for $a \leq x \leq b,\ a \leq y \leq b,$

(b) $\quad \partial^{n-1} G(xy)/\partial x^{n-1},\quad \partial^n G(xy)/\partial x^n$

exist and are continuous for $a \leq x < y,\ y < x \leq b,$

(c) $\quad \partial^{n-1} G(y+, y)/\partial x^{n-1} - \partial^{n-1} G(y-, y)/\partial x^{n-1} = 1/l_n(y),$

(d) $\quad L_x[G(xy)] = N(xy),\quad a \leq x < y,\quad y < x \leq b,$

(e) $\quad U_p[G(xy)] = 0 \quad (p = 1, \cdots, n),$

(f) $\quad \int_a^b \phi_i(x) G(xy)\, dx = 0 \quad (i = 1, \cdots, k).$

The function $G(xy)$ satisfying these conditions depends both on the given differential system and on the choice of the ϕ_i's and ψ_i's.

THEOREM IV. *The function* $G(xy)$ *satisfying conditions* (27) *is a generalized Green's function for the compatible differential system* (1).

By way of proof, let $u_0(x)$ be defined by

$$u_0(x) = \int_a^b G(xs) f(s)\, ds,$$

where $f(s)$ satisfies condition (4). Upon applying the operators* L, U_i $(i = 1, \cdots, n)$, and taking into account condition (4), we get

$$L(u_0) = f(x),\quad U_p(u_0) = 0 \qquad (p = 1, \cdots, n),$$

which shows that u_0 satisfies the semi-homogeneous system (3). Then every solution of the semi-homogeneous system (3) may be written in the form

$$u(x) = \sum_{i=1}^{k} c_i u_i(x) + \int_a^b G(xs) f(s)\, ds,$$

* All details of differentiation are here omitted.

which demonstrates the theorem.

Also, it is readily seen that $\int_a^b \phi_i(x)u_0(x)dx = 0 \ (i=1, \cdots, k)$.

COROLLARY I. *If* $u_0(x) = \int_a^b G(xs)f(s)ds$, *where* $f(s)$ *is any continuous function, then* $u_0(x)$ *satisfies the system*

$$(28) \qquad \begin{cases} L(u_0) = f(x) + \int_a^b N(xs)f(s)ds, \\ U_p(u_0) = 0 \quad (p=1, \cdots, n), \\ \int_a^b \phi_i u_0 dx = 0 \quad (i=1, \cdots, k). \end{cases}$$

COROLLARY II. *If, for each continuous* $f(x)$, *a solution of system* (28) *is given in the form*

$$u_0(x) = \int_a^b G_0(xs)f(s)ds$$

then

$$G_0(xs) = G(xs).$$

Unless explicitly stated otherwise all generalized Green's functions considered in the remainder of this paper will satisfy conditions of type (27). We shall denote by $H(xy)$ the generalized Green's function for system (2).

THEOREM V. *If* $n > 1$, *for any explicit choice of* ϕ_i *and* ψ_i $(i=1, \cdots, k)$,

$$H(xy) \equiv G(yx).$$

By way of proof,* let ξ and η be any two points in the interval (ab). For convenience suppose $\xi < \eta$. Apply Green's theorem letting $u = G(x\xi)$, and $v = H(x\eta)$, choosing as limits of integration a, $\xi-$, then $\xi+$, $\eta-$, and finally $\eta+$, b. Since the integrand is a continuous function of x, we get

$$\begin{aligned} \int_a^b &\{H(x\eta)L[G(x\xi)] - G(x\xi)M[H(x\eta)]\}\,dx \\ &= [P(G,H)]_a^{\xi-} + [P(G,H)]_{\xi+}^{\eta-} + [P(G,H)]_{\eta+}^{b} \\ &= [P(G,H)]_a^{b} - [P(G,H)]_{\xi-}^{\xi+} - [P(G,H)]_{\eta-}^{\eta+} . \end{aligned}$$

But

* The method of proof used here is the same as that used by M. Bôcher in dealing with the ordinary Green's function; *Les Méthodes de Sturm*, p. 104.

$$\int_a^b H(x\eta)L[G(x\xi)]dx$$

$$= \int_a^b H(x\eta) \frac{\begin{vmatrix} 0 & \psi_1(x) & \cdots & \psi_k(x) \\ v_1(\xi) & \int_a^b v_1(s)\psi_1(s)ds & \cdots & \int_a^b v_1(s)\psi_k(s)ds \\ \cdot & \cdot & \cdots & \cdot \\ v_k(\xi) & \int_a^b v_k(s)\psi_1(s)ds & \cdots & \int_a^b v_k(s)\psi_k(s)ds \end{vmatrix}}{\begin{vmatrix} \int_a^b v_1(s)\psi_1(s)ds & \cdots & \int_a^b v_1(s)\psi_k(s)ds \\ \cdot & \cdots & \cdot \\ \int_a^b v_k(s)\psi_1(s)ds & \cdots & \int_a^b v_k(s)\psi_k(s)ds \end{vmatrix}} dx = 0.$$

Likewise

$$\int_a^b G(x\xi)M[H(x\eta)]dx = 0.$$

Also

$$[P(G,H)]_a^b = \sum_{p=1}^{2n} U_p(G)\, V_{2n+1-p}(H) = 0.$$

And since all derivatives of G and H of lower order than $(n-1)$ are continuous functions for all values of x

$$[P(G,H)]_{\xi-}^{\xi+}$$

$$= \left[l_n(x) \left\{ \frac{\partial^{n-1}}{\partial x^{n-1}} G(x\xi)H(x\eta) + (-1)^{n-1}G(x\xi) \frac{\partial^{n-1}}{\partial x^{n-1}} H(x\eta) \right\} \right]_{\xi-}^{\xi+} = H(\xi\eta),$$

$$[P(G,H)]_{\eta-}^{\eta+}$$

$$= \left[l_n(x) \left\{ \frac{\partial^{n-1}}{\partial x^{n-1}} G(x\xi)H(x\eta) + (-1)^{n-1}G(x\xi) \frac{\partial^{n-1}}{\partial x^{n-1}} H(x\eta) \right\} \right]_{\eta-}^{\eta}$$

$$= l_n(\eta) \{(-1)^{n-1}G(\eta\xi)(-1)^n/l_n(\eta)\} = -G(\eta\xi).$$

It follows that

$$H(\xi\eta) = G(\eta\xi) \qquad \xi < \eta.$$

By similar proof

$$H(\xi\eta) = G(\eta\xi) \qquad \xi > \eta.$$

Then from continuity

$$H(\xi\eta) = G(\eta\xi) \qquad \xi = \eta.$$

Hence, $$H(xy) \equiv G(yx).$$

COROLLARY I. *If* $n = 1$, $H(xy) = G(yx) \quad x \neq y.$

COROLLARY II. *For any chosen set of* ϕ_i*'s and* ψ_i*'s the function satisfying conditions* (27) *is unique.*

Since it has been assumed that the determinants

$$\begin{vmatrix} \int_a^b v_1(s)\psi_1(s)\,ds \cdots \int_a^b v_1(s)\psi_k(s)\,ds \\ \cdots\cdots\cdots\cdots\cdots \\ \int_a^b v_k(s)\psi_1(s)\,ds \cdots \int_a^b v_k(s)\psi_k(s)\,ds \end{vmatrix}, \quad \begin{vmatrix} \int_a^b u_1(s)\phi_1(s)\,ds \cdots \int_a^b u_1(s)\phi_k(s)\,ds \\ \cdots\cdots\cdots\cdots\cdots \\ \int_a^b u_k(s)\phi_1(s)\,ds \cdots \int_a^b u_k(s)\phi_k(s)\,ds \end{vmatrix}$$

do not vanish the ψ_i's and the ϕ_i's may be respectively transformed linearly into sets biorthogonal with the v_i's and u_i's $(i = 1, \cdots, k)$. If this is done some of the preceding formulas will simplify considerably. The function N is then given by the formula

$$N\begin{pmatrix} v_1, \cdots, v_k \\ \psi_1, \cdots, \psi_k \end{pmatrix} = -\sum_{i=1}^{k} \psi_i v_i,$$

and $G(xy)$ will satisfy the equation

$$L_x[G(xy)] = -\sum_{i=1}^{k} \psi_i(x) v_i(y).$$

As a special example, consider the system

$$\begin{aligned} u'' &= 0, \\ u(0) + u(1) &= 0, \\ u'(0) - u'(1) &= 0. \end{aligned}$$

This system is compatible of order one, and a solution is $u_1(x) = 1 - 2x$. Its adjoint system is

$$\begin{aligned} v'' &= 0, \\ v(0) - v(1) &= 0, \\ v'(0) + v'(1) &= 0, \end{aligned}$$

which has a solution $v_1(x) = 1$.

If $\phi_1(x) = \bar{u}_1(x) = 1 - 2x$, and $\psi_1(x) = \overline{v}_1(x) = 1$, the generalized Green's functions for the two systems are respectively

$$\begin{aligned} G_1(xy) &= -x^2/2 - x/2 - 2xy^3 + 3xy^2 + y^3 - 3y^2/2 + y/2, && x \leq y, \\ &= -x^2/2 + x/2 - 2xy^3 + 3xy^2 + y^3 - 3y^2/2 - y/2, && x \geq y. \\ H_1(xy) &= -3x^2/2 + 3x^2y + x^3 - 2x^3y - y^2/2 + y/2 - x/2, && x \leq y, \\ &= -3x^2/2 + 3x^2y + x^3 - 2x^3y - y^2/2 - y/2 + x/2, && x \geq y. \end{aligned}$$

If $\phi_1(x) = \psi_1(x) = x$, the generalized Green's functions become

$$\begin{aligned} G_2(xy) &= -x^3/3 + 3xy - 2xy^3 + y^3 - 6x/5 - y + 4/15, && x \leq y, \\ &= -x^3/3 + 3xy - 2xy^3 + y^3 - x/5 - 2y + 4/15, && x \geq y. \\ H_2(xy) &= x^3 - 2x^3y + 3xy - y^3/3 - y/5 - 2x + 4/15, && x \leq y, \\ &= x^3 - 2x^3y + 3xy - y^3/3 - 6y/5 - x + 4/15, && x \geq y. \end{aligned}$$

If we choose $\phi_1(x) = x^2$, and $\psi_1(x) = x$, the corresponding generalized Green's functions are

$$\begin{aligned} G_3(xy) &= -x^3/3 - xy^4 + 2xy + y^4/2 - x - y/2 + 1/6, && x \leq y, \\ &= -x^3/3 - xy^4 + 2xy + y^4/2 - 3y/2 + 1/6, && x \geq y. \\ H_3(xy) &= x^4/2 - x^4y + 2xy - y^3/3 - 3x/2 + 1/6, && x \leq y, \\ &= x^4/2 - x^4y + 2xy - y^3/3 - x/2 - y + 1/6, && x \geq y. \end{aligned}$$

An Expansion Involving p Inseparable Parameters Associated with a Partial Differential Equation.*

By Chester C. Camp.

1. *Introduction and Formal Setting.* The partial differential equation

$$\sum \partial u/\partial x_l + (\lambda h + k)u = 0,\dagger \tag{1}$$

may be solved by Lagrange's method provided h and k each consist of integrable functions of the several single variables and polynomials in more than one variable.‡ It is clear that one may then solve (1) in the form

$$u = e^{-\lambda H - K} u_1 \tag{2}$$

where $h = DH + \sum a_l(x_l)$, $k = DK + \sum g_l(x_l)$, $D \equiv \sum \partial/\partial x_l$

and u_1 is a solution of the equation

$$\sum [\partial u_1/\partial x_l + (\lambda a_l + g_l)u_1] = 0. \tag{3}$$

It may be shown that the general solution of (3) is

$$u_1 = exp\ [-\lambda \sum A_l(x_l) - \sum G_l(x_l)] \cdot \phi(x_1 - x_2, x_1 - x_3, \cdots, x_1 - x_p), \tag{4}$$

where $A_l(x_l) = \int a_l(x_l)dx_l$ and $G_l(x_l) = \int g_l(x_l)dx_l$,

and l runs from 1 to p.

If ϕ is so chosen that $u_1 = \prod X_l(x_l)$, equation (3) leads to the differential system

$$\begin{aligned} &X_1' + (\lambda a_1 + g_1 - \textstyle\sum' \mu_l)X_1 = 0. \\ &X_l' + (\lambda a_l + g_l + \mu_l)X_l = 0, \qquad (l = 2, 3, \cdots, p). \end{aligned} \tag{5}$$

---

* Presented to the Society, Sept. 9, 1926.

† When l runs from 1 to p the sum is indicated by Σ and the product by Π, when l runs from 2 to p the sum is indicated by Σ', when l runs from 3 to p the sum is indicated by Σ''.

‡ They might contain such terms as $x_1^2 x_2^3 \sin x_3$, which may be integrated as to x_1 under the condition $dx_1 = dx_2 = \cdots = dx_p$ by first writing

$$\int x_1^2 x_2^3 \sin x_3 dx_3 = -x_1^2 x_2^3 \cos x_3 + \int (3x_1^2 x_2^2 + 2x_1 x_2^3) \cos x_3 dx_3.$$

Further integrations by parts diminish the degrees of the multipliers of $\cos x_3$ or $\sin x_3$ to zero.

In case H and K each satisfy the boundary conditions

$$(6)\quad [u(x_1, x_2, \cdots, x_p)]_{x_l=\pi} = [u(x_1, x_2, \cdots, x_p)]_{x_l=-\pi} \quad (l = 1, 2, \cdots, p)$$

they will not affect the principal parameter values of λ for the system (1), (6); namely

$$(7)\qquad \lambda_{m_1, m_2, \ldots, m_p} = \sum (m_l i - G_l)/\sum A_l$$

where m_l is an integer, $2\pi G_l = G_l(\pi) - G_l(-\pi)$, and $2\pi A_l = A_l(\pi) - A(-\pi)$, $(l = 1, 2, \cdots, p)$.

Let us assume that each $a_l(x_l)$ is either identically zero or of constant sign, also that $\sum A_l \neq 0$. The case in which every $g_l(x_l) \equiv 0$ will be considered first since the proof of convergence is thereby simplified; secondly, the case in which some $g_j(x_j) \not\equiv 0$; and finally that in which h and k may each contain functions of more than one variable.

It is to be noticed that the system (5) involves parameters which are tangled up so to speak; i. e., it is not possible to separate them so as to have p ordinary differential equations each involving a single parameter.

One may readily show that the adjoint system with similar boundary conditions has the same principal parameter values; e. g., for the simplest case the adjoint system

$$(8)\qquad \begin{aligned} &-Y_1' + (\lambda a_1 - \textstyle\sum' \mu_l) Y_1 = 0, \\ &-Y_l' + (\lambda a_l + \mu_l) Y_l = 0, \qquad (l = 2, 3, \cdots, p) \end{aligned}$$

$$(9)\qquad Y_l(\pi) = Y_l(-\pi), \qquad (l = 1, 2, \cdots, p)$$

has the characteristic values

$$(10)\qquad \begin{aligned} &\lambda^* = \sum m_l i / \sum A_l \\ &\mu_l = \lambda^* (\textstyle\sum A_l - A_l) - (\sum m_l - m_l) i = -\lambda^* A_l + m_l i, \\ &\qquad\qquad (l = 2, 3, \cdots, p). \end{aligned}$$

By using two sets of characteristic values which differ in at least one of the parameters we may adapt (5) to one and (8) to the other using a double asterisk to indicate the solutions associated with the second set, thus

$$(11)\qquad \begin{aligned} &X_1'^* + (\lambda^* a_1 - \textstyle\sum' \mu_l^*) X_1^* = 0, \\ &X_l'^* + (\lambda^* a_l + \mu_l^*) X_l^* = 0, \qquad (l = 2, 3, \cdots, p); \end{aligned}$$

$$(12)\qquad \begin{aligned} &-Y_1'^{**} + (\lambda^{**} a_1 - \textstyle\sum' \mu_l^{**}) Y_1^{**} = 0, \\ &-Y_l'^{**} + (\lambda^{**} a_l + \mu_l^{**}) Y_l^{**} = 0, \quad (l = 2, 3, \cdots, p). \end{aligned}$$

Multiplying the jth of equations (11), (12) by Y_j^{**}, $-X_j^*$, respect-

ively, adding, integrating from $-\pi$ to π and using (9) and $X_j^*(\pi) = X_j^*(-\pi)$, one obtains

$$(13)\qquad (\lambda^* - \lambda^{**}) \int_{-\pi}^{\pi} a_1 X_1^* Y_1^{**} dx_1 - \Sigma' (\mu_l^* - \mu_l^{**}) \int_{-\pi}^{\pi} X_1^* Y_1^{**} dx_1 = 0,$$

$$(\lambda^* - \lambda^{**}) \int_{-\pi}^{\pi} a_j X_j^* Y_j^{**} dx_j + (\mu_j^* - \mu_j^{**}) \cdot \int_{-\pi}^{\pi} X_j^* Y_j^{**} dx_j = 0, \quad (j = 2, 3, \cdots, p).$$

Since the sets of characteristic values are not identical, it is necessary that the determinant of (13) vanish. This gives the conjugacy condition

$$(14)\qquad \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} (\Sigma a_l) \prod X_l^* Y_l^{**} dx_l = 0.$$

If one assumes the expansion

$$(15)\qquad f(x_1, x_2, \cdots, x_p) = \sum_{\substack{m_l=-\infty \\ l=1, 2, \ldots, p}}^{\infty} C_{m_1, m_2, \ldots, m_p} u_1^*(x; m_1, m_2, \cdots, m_p)$$

where f is made up of a finite number of pieces, each real, continuous, and possessing a continuous partial derivative in each variable, and the argument x in the principal solution u_1^* is written in place of $x_1, x_2, \cdots, x_p$, then formally † if $v_1^* \equiv \prod Y_l^*(x_l)$,

$$C_{m_1, m_2, \ldots, m_p} = \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} f(s_1, \cdots, s_p) \Sigma a_l(s_l) v_1^*(s; m_1, \cdots, m_p) \cdot \prod ds_l/(2\pi)^p \Sigma A_l.$$

Let us show that the formal series actually converges to the so-called mean value of f.

2. *Convergence Proof.* Case I, $g_l(x_l) \equiv 0$, $(l = 1, 2, \cdots, p)$. It is important in setting up the analog of the Green's function to make a transformation of parameters as follows:

$$(16)\qquad \begin{cases} \nu_1 = \lambda A_1 - \Sigma' \mu_l \\ \nu_l = \lambda A_l + \mu_l, \qquad (l = 2, 3, \cdots, p). \end{cases}$$

One can then readily prove that one term of the series (15) is the residue

† Cf. an article by Carmichael, "Boundary Value and Expansion Problems," *American Journal of Mathematics*, Vol. 43, No. 4 (Oct. 1921), p. 260.

as to $\nu_1, \nu_2, \cdots, \nu_p$ at a set of values corresponding to a set of characteristic values for $\lambda, \mu_2, \mu_3, \cdots, \mu_p$, of the function

$$(17)\quad \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi} f(s_1,\cdots,s_p)\ \Sigma a_l(s_l)G(x;s;\lambda,\mu_2,\cdots,\mu_p)\cdot\prod ds_l$$

where

$$(18)\quad G\equiv\exp\ [\int_{x_1}^{s_1}(\lambda a_1-\Sigma'\mu_l)dx_1 + \Sigma\int_{x_l}^{s_l}(\lambda a_l+\mu_l)dx_l]/(\Sigma A_l)\prod(1-e^{-2\pi\nu_l}),\quad s_l<x_l);$$

and when $s_j > x_j$ for one or more values of j the factor $(1-e^{-2\pi\nu_j})$ in the denominator for each such j is to be replaced by $(e^{2\pi\nu_j}-1)$.

The convergence is studied by the powerful and elegant method of Birkhoff adapted to multiple series by which one takes the limit of a p-fold contour integral I whose curves, whether circles or rectangles in the several ν-planes, enclose an indefinitely increasing number of characteristic values. One is led therefore to investigate one of the 2^p pieces in the form

$$(19)\quad I_{-\pi,-\pi,\cdots,-\pi}^{x_1,x_2,\cdots,x_p} = \int_{\Gamma_p}\int_{\Gamma_{p-1}}\cdots\int_{\Gamma_1}\int_{-\pi}^{s_1}\cdots\int_{-\pi}^{s_p} f(\sum_{l=1}^{p}a_l)\,G\prod ds_l d\nu_l/(2\pi i)^p.$$

Replace Σa_l by

$$(\Sigma A_l)\{[\lambda a_1-\Sigma'\mu_l]+\Sigma'[\lambda a_l+\mu_l]\}/\Sigma\nu_l$$

thus separating the $2p$-fold integral into p pieces. If one integrates each by parts using the brackets in order to integrate exponentials, one obtains as the integrands of the contour integrals, exclusive of the factor

$$1/(2\pi i)^p\,(\Sigma\nu_l)\prod(1-e^{-2\pi\nu_l}),$$

$$(20)\quad \int_{-\pi}^{x_p}\int_{-\pi}^{x_{p-1}}\cdots\int_{-\pi}^{x_2}\exp\ [\Sigma'\int_{x_l}^{s_l}(\lambda a_l+\mu_l)dx_l].\ \{f(x_1-0,s_2,\cdots,s_p)-f(-\pi,s_2,\cdots,s_p)\cdot \exp\ [-\int_{-\pi}^{s_1}(\lambda a_1-\Sigma'\mu_l)dx_1]\}dx_2\cdots dx_p$$

$$+\sum_{j=2}^{p}\int_{-\pi}^{x_p}\int_{-\pi}^{x_{p-1}}\cdots\int_{-\pi}^{x_{j+1}}\int_{-\pi}^{x_{j-1}}\cdots\int_{-\pi}^{x_1}$$

$$\exp\ [\sum_{l\neq j}'\int_{x_l}^{s_l}(\lambda a_l+\mu_l)dx_l+\int_{x_1}^{s_1}(\lambda a_1-\Sigma'\mu_l)dx_1]\cdot [f(s_1,\cdots,s_p)\exp\int_{x_j}^{s_j}(\lambda a_j+\mu_j)dx_j]_{-\pi}^{x_j}\prod_{l=1}^{p}ds_l/ds_j$$

$$-\int_{-\pi}^{x_p}\cdots\int_{-\pi}^{s_1}u_1(x_1,\cdots,x_p)\,v_1(s_1,\cdots,s_p)\,(\Sigma\partial f/\partial s_l)\prod ds_l+\epsilon.$$

Here $\epsilon = 0$ if f is continuous. In case of a discontinuity, in x_1 for example, there will be an extra term in the integration by parts, which would involve a factor $S \exp\left[-\int_x^{x_1} (\lambda a_1 - \Sigma \mu_l)\, dx_1\right]$ in lieu of

$$f(-\pi, s_2, \cdots, s_p) \exp\left[-\int_{-\pi}^{x_1} (\lambda a_1 - \Sigma \mu_l)\, dx_1\right]$$

where S is the saltus or fluctuation of f in x_1 at the point $x_1 = x$. By a subsequent argument such a piece contributes zero to the limit. The analogous terms are not mentioned explicitly in a previous paper by the author * but are included by the notation $O(1/\lambda^2)$ or $O(1/\lambda)$, whereas such terms are not considered in a paper by Carman † although they can as readily be disposed of there.

The integrands of the exponentials are now written in the form $\lambda(a_l - A_l) + \nu_l$, $(l = 1, 2, \cdots, p)$, and one integrates $p - 1$ times by parts (each time with respect to a different variable), using the exponentials $\exp[\nu_l(s_l - x_l)]$, each integral of (20) previously obtained as an upper limit in the first integration by parts. One item of the result is

$$(21) \qquad \frac{1}{(2\pi i)^p} \int_{\Gamma_p} \cdots \int_{\Gamma_1} \frac{f(x_1 - 0, x_2 - 0, \cdots, x_p - 0)}{\prod \nu_l (1 - e^{-2\pi\nu_l})} \prod d\nu_l$$

Which is equivalent in the limit as $|\nu_l|$ becomes infinite to

$$f(x_1 - 0, x_2 - 0, \cdots, x_p - 0)/2^p,$$

as is seen by using circular contours with centers at the origins of the complex planes and radii $\frac{1}{2} + m_i$, where m_i is a positive integer. This is also a consequence of the following lemma.

LEMMA: ‡ If $0 < h < 1$, then $\lim_{|z|\to\infty} \frac{1}{2\pi i} \int_{C_{|z|}} \left| \frac{e^{-2\pi h z} dz}{z(1 - e^{-2\pi z})} \right| = 0 =$

$$\lim_{|z|\to\infty} \frac{1}{2\pi i} \int_{C_{|z|}} \left| \frac{e^{2\pi h z} dz}{z(e^{2\pi h z} - 1)} \right|$$

where $C_{|z|}$ represents a circle in the z-plane with center at the origin and

---

* "Multiple Fourier Series Expansions," *Transactions of the American Mathematical Society*, Vol. 25 (1923), pp. 123-134.

† "Convergence of Fourier Series," *Bulletin of the American Mathematical Society* (Oct. 1924).

‡ For details of proof one may refer to the papers by Carman and the author cited above, or to an article by Birkhoff and Langer, "Boundary Problems and Developments Associated with a System of Ordinary Linear Differential Equations of the First Order," *Proceedings of the American Academy*, Vol. 58, No. 2 (Apr. 1923), pp. 112-114.

radius $|z|$. A vertical rectangle with center at the origin and increasing altitude may be substituted for the circle. For $h=0$, or $h=1$ the limits of the first integral are $\frac{1}{2}$, or $-\frac{1}{2}$, respectively.

Use is made of this to show also that the contribution of each of the other integrals to the limit is zero.

Since the characteristic values are all pure imaginaries it will be sufficient to employ rectangular contours of distinct widths δ_i held fixed. The limit as each altitude becomes infinite represents the sum of an infinite series. This consists of the sum of a series of residues and hence the same sum will be obtained by using circular contours in which the several radii $|\nu_l|$ become infinite. Let the radii be so chosen that each $|\nu_l|$ and $|\sum \nu_l|$ become infinite together. The lengths of the rectangles may be chosen so as to include the same principal parameter values as the circular contours, the distance of the contours in either case from the nearest characteristic values being kept bounded below.

The third contour integral of (20) may be shown to converge in the limit to zero. To prove this one compares the infinite series represented by it with the simpler series obtained by removing according to the second mean value theorem the exponentials of the form $|\exp\lambda\int_{\alpha_l}^{s_l}(a_l-A_l)\,dx_l|$. If $|\sum \partial f/\partial s_l|$ is now replaced by an upper bound M, the remaining integral can readily be shown by direct integration and the above lemma to contribute zero.

The product of the exponentials is bounded uniformly over the rectangles suggested above and hence the original series converges to zero for any interior point. For points on the boundary the integral still converges to zero on account of the factor $\sum \nu_l$ in the denominator, which becomes infinite for circular contours.

To show that the parts of (20) involving the lower limits converge to zero for an interior point one compares the series they represent with simpler series. As an example, in the first one, we remove the exponential factor whose index is $\Sigma'\int_{\alpha_l}^{s_l}\lambda(a_l-A_l)\,dx_l-\lambda\int_{-\pi}^{\alpha_1}(a_1-A_1)\,dx_1$, take absolute values, and replace $f(-\pi, s_2, \cdots, s_p)$ by an upper bound. The resulting contour integral converges by the lemma and the fact that $|\sum \nu_l|\to\infty$ when $|\nu_1|\to\infty$. The original series will converge to zero and hence its contour integral will also over the rectangular or circular contours selected above.

The same argument holds for the remaining integrals of (20) and the

analogous ones obtained in the $p-1$ successive integrations by parts. In the first of these one obtains the form

$$(22)\quad \frac{-1}{(2\pi i)^p}\int_{\Gamma_p}\cdots\int_{\Gamma_1}\int_{-\pi}^{x_p}\cdots\int_{-\pi}^{x_2}\exp\,[\nu_2(s_2-x_2)+$$
$$\Sigma''\int_{x_l}^{s_l}(\lambda a_l+\mu_l)\,dx_l]\cdot\partial/\partial s_2\,\{f(x_1-0,\,s_2,\cdots,s_p)\cdot$$
$$\exp\int_{x_2}^{s_2}\lambda(a_2-A_2)dx_2\}\prod ds_l d\nu_l/N_2(\nu)ds_1,$$

where $N_2(\nu)\equiv\nu_2(\Sigma\nu_l)\prod[1-\exp(-2\pi\nu_l)]$.

For this consider a comparison series whose contour integral is

$$(23)\quad \frac{-1}{(2\pi i)^p}\int_{\Gamma_p}\cdots\int_{\Gamma_1}\int|\exp\nu_2(s'_2-x_2)|\cdot$$
$$\int_{-\pi}^{x_p}\cdots\int_{-\pi}^{x_2}|\exp[-\nu_1A_2(s_2-x_2)/\Sigma A_l+\Sigma'''\nu_l(s_l-x_l)]$$
$$[\partial f/\partial s_2+\lambda(a_2-A_2)f]\,|\prod ds_l\,|\,d\nu_l\,|/|\,N_2(\nu)ds_1$$

where $-\pi<s_2'\leqq x_2$ by the mean value theorem, and where certain exponential factors which are bounded on the rectangular contours have been removed. But $s_2'\neq x_2$ unless ν_2 is on the axis of imaginaries. In such a case one omits from the contour Γ_2 small arcs over which the contour integral will contribute ϵ.

Since the last bracket of (23) divided by $\Sigma\nu_l$ is bounded, it is clear that the limit of (23) over a circle in the ν_2-plane and rectangular contours in the others will be zero.

The integral (22) is seen by the previous reasoning to contribute zero also. A like method applies to the other integrals of this form and to analogous forms in the other integrations by parts. The simpler comparison series converge to zero. One may change the one circular contour to a rectangle. The exponential multipliers in each case are bounded uniformly on the rectangular contours. Hence the original series converge to zero in the limit.

In case $A_2=a_2=0$ in (23) one of the exponents vanishes and the factor $\Sigma\nu_l$ is used instead of ν_1 in the denominator of the contour integral.

For an interior point each of the other parts of I may be shown to contribute analogous results, so that the whole integral has the limit $\Sigma f(x_1\pm 0,\,x_2\pm 0,\cdots,x_p\pm 0)/2^p$, in other words the so-called mean value of f at the point $(x_1,\,x_2,\cdots,x_p)$. For points on the boundary the series will

still converge to the mean value if we regard f as periodic in each argument with period 2π.

Case II, some $g_j(x_j) \not\equiv 0$. The presence of $\sum g_l(x_l)$ as a multiplier of u_1 gives for characteristic values

$$(24) \qquad \begin{cases} \lambda^* = \sum (m_l i - G_l)/\sum A_l, \\ \mu_l^* = m_l i - \lambda^* A_l - G_l, \quad (l = 2, 3, \cdots, p). \end{cases}$$

The transformation of parmeters analogous to (16) is

$$(25) \qquad \begin{cases} v_1 = \lambda A_1 - \sum' \mu_l + G_1, \\ v_l = \lambda A_l + \mu_l + G_l, \quad (l = 2, 3, \cdots, p). \end{cases}$$

$$(26) \qquad u_1 = \exp\{-\sum[\lambda A_l(x_l) + G_l(x_l)] + \sum' \mu_l(x_1 - x_l)\} = 1/v_1,$$

hence the analog of (19) will contain the extra factor $\exp \sum [G_l(s_l) - G_l(x_l)]$ in the integrand. Since each g_l is integrable $G_l(x_l)$ is continuous. It is clear that the f with the multiplier $\exp \sum G_l(s_l)$ may be regarded as a new f and that for any interior point the sum of the series analogous to that in (15) will be the same mean value as before.

Case III, DH or $DK \not\equiv 0$. For the more general form

$$(27) \qquad \sum \partial u/\partial x_l + \{\lambda[\sum a_l(x_l) + DH(x_1, \cdots, x_p)] + \sum g_l(x_l) + DK(x_1, \cdots, x_p)\}u = 0$$

and the same boundary conditions (6) the general solution is given by (2) and (4). Let us first choose u_1 in the form $\prod X_l(x_l)$ as in (26). It is easy to prove that there will still be suitable values for λ^* in case H and K each satisfy (6). Then the presence of DH and DK will not alter the characteristic values. The conjugacy condition is now that in which $\sum a_l(s_l)$ is augmented by $DH(s_1, s_2, \cdots, s_p)$.

The denominator of the Green's function is not altered since

$$(28) \qquad \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} DH \prod ds_l = 0.$$

In the convergence proof one has the extra factor

$$(29) \qquad \exp\{\lambda[H(s_1, s_2, \cdots, s_p) - H(x_1, \cdots, x_p)] + K(s_1, \cdots, s_p) - K(x_1, \cdots, x_p)\}$$

in the integrand, besides the new factor of the conjugacy condition. By separ-

ating the parts of $DH(s_1, \cdots, s_p)$ and integrating by parts one clearly obtains for one part of $I^{x_1, \cdots, x_p}_{-\pi, \cdots, -\pi}$ the same element of the mean value as before, since for $s_l = x_l$, $(l = 1, 2, \cdots, p)$ the value of (29) is unity. To show that the other pieces contribute zero for any interior point it is sufficient to use as a comparison series that formed by multiplying the several terms of the series for Case II by (29) and then to take absolute values. This factor is bounded uniformly on the rectangular contours for increasing altitudes. Hence the limit is zero. On the boundaries, if f is not regarded as periodic, certain parts of the mean value will require multipliers which are functions of the arguments that vary on the particular boundary.

Finally, it is possible to multiply the solution u_1 as given in (26) by a function ϕ with arguments as in (4) provided ϕ is periodic of period 2π in each. We take $v = 1/u$, then the contour integral will have the additional factor $\phi(x_1 - x_2, \cdots, x_1 - x_p)/\phi(s_1 - s_2, \cdots, s_1 - s_p)$ where f/ϕ may again be looked upon as a new f. If ϕ does not vanish obviously the series will again converge to the mean value at any interior point. We may therefore state the most general result in the following

THEOREM: *Let $f(x_1, x_2, \cdots, x_p)$ be made up of a finite number of pieces, each real, continuous, and possessing a continuous partial derivative in each argument, and let H, K and ϕ satisfy* (6), *while DH, DK consist of polynomials in the p variables or polynomials each multiplied by an integrable function of a single variable, $g_l(x_l)$ is integrable, each $a_l(x_l)$ is integrable and either identically zero or of constant sign, $-\pi \leqq x_l \leqq \pi$, but $\sum \int_{-\pi}^{\pi} a_l(x_l)\,dx_l \neq 0$, also $\phi \neq 0$, and $\partial\phi/\partial x_l$ is continuous; then the expansion $\sum_{\substack{m_i=-\infty \\ i=1, 2, \ldots, p}}^{\infty} C_{m_1, \ldots, m_p}\ u^*(x; m_1, \cdots, m_p)$, where u^* is the most general characteristic solution of* (27), (6), $v^* = 1/u^*$, *and*

$$C_{m_1, m_2, \ldots, m_p} = \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} fv^*(\sum a_l + DH) \prod ds_l / \sum A_l, \tag{30}$$

converges to the so-called mean value of f at any interior point. In any interior closed region in which f and its first partial derivatives are continuous the series converges uniformly to f.

It is evident that the generalization involving ϕ can be applied to the multiple-Birkhoff series † and to other expansions associated with partial differential equations and boundary conditions.

† Cf. *Transactions of the American Mathematical Society*, Vol. 25, No. 3, pp. 338-342.

Moreover, it is believed that the convergence proof by the contour integral method for all such series can be carried through only in case the number of independent parameters equals the number of independent variables employed. For example, if in (4) ϕ had been chosen as unity the characteristic values (7) of λ would be simply $(mi - \sum G_l)/\sum A_l$ and the corresponding expansion for f would involve but one parameter. Another such expansion is given by Carmichael * for $f(x, y)$ with just one parameter λ. It is desirable to investigate further whether either of these formal expansions is valid.

UNIVERSITY OF NEBRASKA,
LINCOLN, NEBR.

* *American Journal of Mathematics, loc. cit.*, p. 259.

On the Asymptotic Evaluation of Functions Defined by Contour Integrals.

By D. M. Wrinch.

Introduction.

1. In the present paper, the author attempts to give unity to the investigations of the asymptotic expansions for large values of n, of functions defined by the integral

$$I = \int e^{nf(z)} F(z)\, dz$$

taken along paths in the z-plane. The problem of grouping paths in such a way that the asymptotic expansions of the functions, defined by the integral I along paths belonging to the same group are identical, is first considered. The asymptotic evaluation of the integral I, along any one of a large class of paths is then effected by means of the discussion of the value of the integral I along certain standard paths.

This procedure is precisely similar to that adopted in the theory of complex integration. Cauchy's theorem gives specifications * of a group of paths associated with two points, such that the value of the integral

$$\int_C F(z)\, dz$$

is the same, whichever path of the group C may be. Next, the value of the integral taken along any one of a wide class of paths may be obtained by the discussion of the value of the integral taken along certain standard paths. The calculus of residues belongs to this part of the subject.

* Thus, e. g. Riemann considered the class of paths running from a point z_1 to a second point z_2 which is such that at all points on each path, P and Q, the real and imaginary parts of $F(z)$ have continuous derivates with respect to x and y such that

$$\partial P/\partial x = \partial Q/\partial y, \quad \partial Q/\partial x = -\partial P/\partial y.$$

Again, in Goursat's form of Cauchy's theorem, each member of the class of paths under consideration runs from a point z_1 to a second point z_2, and at all points on each path and at all points between any two of the paths, $F'(z)$ exists. Either of these specifications yields a group of paths associated with two points which is such that the value of the integral

$$\int_C F(z)\, dz$$

is the same whichever path of the group C may be. Cp. Watson, 'Complex Integration and Cauchy's Theorem,' *Cambridge Tracts in Mathematics*, No. 15.

In Cauchy's theorem, the paths belonging to one group, all begin at one point and end at a second point. In the corresponding theorem of asymptotic evaluation, it is sufficient to specify a certain *continuum* in which the paths belonging to one group must start and a certain *continuum* in which they must end. The continua in question are of a special type; they are called, in this paper, 'valleys' or 'valley continua.' An 'M-valley' associated with the function $f(z)$ is a continuum in which the real part of $f(z)$ has M as its upper bound.* Under certain circumstances, if m_1 and m_2 are two M-valleys associated with the function $f(z)$, and if C is a path starting in the continuum m_1 and ending in the continuum m_2, the asymptotic value of the integral

$$e^{-nM} \int_C e^{nf(z)} F(z)\, dz$$

is the same whichever path C may be. The results are obtained by considering the deductions which can be drawn from the behaviour of the real part of the function $f(z)$ on the path C, as to the asymptotic value, for large values of n, of the integral I taken along the path C. Although, on occasion it has been pointed out † that the integral I taken along a certain path differs from the integral taken along a second path C' by a term of order $e^{-n\delta}$, where δ is some positive number, the author is unaware of any previous attempt to study the general problem of grouping paths in such a way, that the functions defined by the integral I taken along paths belonging to one group admit the same asymptotic expansion for large values of n.

2. The asymptotic evaluation of the integral

$$\int e^{nf(z)} F(z)\, dz$$

has been discussed in a number of particular cases. Thus the analysis necessary for the Bessel functions defined by means of the integral

$$\int e^{-iN \sin z} e^{ikz}\, dz \qquad (N = ne^{i\nu})$$

taken along certain paths, was developed by Debye ‡ and supplemented by Watson.§ The analysis for the various functions defined by the integral

$$\int e^{i(z^3 - Nz)}\, dz \qquad (N = ne^{i\nu})$$

---

* The idea of a region (not necessarily a continuum) in which the real part of $f(z)$ is less than a specific number M has been used by Debye, Brillouin, Watson, and Perron. *V. inf.*

† Cp. Perron, *Münchener Berichte* (1917), pp. 191-220.

‡ *Mathematische Annalen* (1909), Vol. 67, pp. 535-568; *Münchener Berichte* (1910) (5).

§ *Proceedings of the London Mathematical Society* (1916) (2), Vol. 17, pp. 116-148. Cp. also *Quarterly Journal* (1917), Vol. 48, pp. 1-18; *Transactions of the Cambridge Philosophical Society* (1918), Vol. 22, pp. 277-308.

along certain paths, and for various other functions defined by integrals which have a polynomial in the exponent of the integrand (e. g. by the integral

$$\int e^{\frac{1}{4}(z^4-pz^2-qz)}\,dz)$$

was set up by Brillouin.* The analysis for the function $W_{k,m}(N)$ defined in terms of the integral

$$\int(-z)^{-k-\frac{1}{2}+m}(1+z/N)^{k-\frac{1}{2}+m}\,e^{-z}\,dz \qquad (N=ne^{i\nu})$$

was given by Watson.† These various writers called the method employed the Method of Steepest Descents,‡ the Sattelpunkte Methode,§ and the Méthode du Col.* In 1917, Perron's † important paper appeared. It dealt with the asymptotic evaluation of the integral

$$\int e^{nf(z)}\,F(z)\,dz$$

along certain types of paths. The asymptotic evaluation of the integral I along the standard paths adopted in this paper is derived from Perron's analysis. Perron's analysis is, however, carried a step further.

1. Definitions and Preliminary Considerations.

1.1. Let

$$I_C(n) = \int_C e^{nf(z)}\,F(z)\,dz,$$

C being a path in the z-plane. The absolute value of the function $I_C(n)$ is evidently dependent upon the behaviour of the real part of $f(z)$ on the path C. Suppose that there exists a positive number δ such that

$$\Re f(z) \leq M-\delta$$

on the path C. Then, assuming that the integral $I_C(n)$ exists, at least for sufficiently large values of n, we may deduce that

$$\begin{aligned} |I_C(n)| &= \left|\int_C e^{(n-b)f(z)}\,e^{bf(z)}F(z)\,dz\right| \\ &\leq e^{(n-b)(M-\delta)}\left|\int_C e^{bf(z)}F(z)\,dz\right| \qquad (n\geq b) \\ &\leq K\,e^{n(M-\delta)} \end{aligned}$$

where K is independent of n. Thus

$$I_C(n)\,e^{-nM} = O(e^{-n\delta})$$

---

* *Annales de l'École Normale Supérieure* (1916) (3), Vol. 33, pp. 17-69.
† *Loc. cit.* ‡ See footnote § of p. 270. § See footnote ‡ of p. 270.

where $\delta > 0$.

Suppose now that there are two paths C_a and C_b starting at the same point and that a third path C_r exists which joins the end points of C_a and C_b and is specified with respect to the integral I in such a way that

$$\int_{C_b} e^{nf(z)} F(z)\, dz = \left(\int_{C_a} + \int_{C_r} \right) e^{nf(z)} F(z)\, dz;$$

then, if there exists a positive number δ such that on the path C_r

$$\Re f(z) \leq M - \delta$$

it follows that

$$e^{-nM} \int_{C_b} e^{nf(z)} F(z)\, dz - e^{-nM} \int_{C_a} e^{nf(z)} F(z)\, dz = O(e^{-n\delta})$$

it being assumed that the integral

$$\int e^{nf(z)} F(z)\, dz$$

taken along the paths C_a, C_b and C_r exists, at least for sufficiently large values of n.

Again, suppose that there are two paths C_a and C_b and that there exist paths C_r and C_s such that

$$\int_{C_b} e^{nf(z)} F(z)\, dz = \left(\int_{C_r} + \int_{C_a} + \int_{C_s} \right) e^{nf(z)} F(z)\, dz.$$

Then, if on C_r and C_s

$$\Re f(z) \leq M - \delta$$

where δ is some positive number, it follows that

$$e^{-nM} \int_{C_b} e^{nf(z)} F(z)\, dz - e^{-nM} \int_{C_a} e^{nf(z)} F(z)\, dz = O(e^{-n\delta}).$$

And here again it has been assumed that the integral along the various paths exists, at least for sufficiently large values of n.

1.2. It will be convenient to introduce the symbol $\sim_n$ to denote the relation between two functions $\phi(n)$ and $\psi(n)$, when there exists a positive number δ such that, for large values of n

$$\phi(n) - \psi(n) = O(e^{-n\delta}).$$

Suppose that $\psi(n)$ admits the asymptotic development *

$$\sum_{m=0}^{\infty} c_m n^{-(a+m)/p}$$

for large values of n, where p is some positive integer, then if

$$\phi(n) \sim_n \psi(n)$$

---

* A function $\chi(n)$ is said to admit the asymptotic development

$$a_0 + a_1 n^{-1/p} + a_2 n^{-2/p} + \cdots$$

for large values of n,

[which is also written $\chi(n) \backsim a_0 + a_1 n^{-1/p} + a_2 n^{-2/p} + \cdots$]

it follows that $\phi(n)$ admits the same asymptotic development.

Further, we shall say that certain paths are 'equipollent with respect to the integral I,' when the value of the integral I along any of the paths is the same.

1.3. The results obtained above may now be restated in the following way:

1. If C_a and C_b are two paths starting at the same point, and if there exists a third path C_r such that C_b and $C_a + C_r$ are equipollent for the integral

$$I = \int e^{nf(z)} F(z) dz,$$

then if on C_r

$$\Re f(z) \leq M - \delta$$

where δ is some positive number, it follows that

$$e^{-nM} \int_{C_b} e^{nf(z)} F(z)\, dz \sim_n e^{-nM} \int_{C_a} e^{nf(z)} F(z)\, dz.$$

2. If C_a and C_b are two paths and there exist two other paths C_r and C_s such that C_b and $C_r + C_a + C_s$ are equipollent for the integral

$$I = \int e^{nf(z)} F(z)\, dz,$$

then, if on C_r and C_s

$$\Re f(z) \leq M - \delta$$

where δ is some positive number, it follows that

$$e^{-nM} \int_{C_b} e^{nf(z)} F(z)\, dz \sim_n e^{-nM} \int_{C_a} e^{nf(z)} F(z)\, dz.$$

1.4. These considerations at once direct attention to the group of paths which have the characteristic that on them, the real part of $f(z)$ is less than some specific number M. We therefore proceed to investigate the regions of the z-plane in which

$$\Re f(z) \leq M - \delta < M$$

for various values of M.

---

if $\lim_{n \to \infty} | n^{k/p} \{\chi(n) - [a_0 + a_1 n^{-1/p} + \cdots + a_{k-1} n^{-k/p}]\} | = 0$

for all values of k. The expression is also used in an extended sense. A function $\chi(n)$ is said to admit the asymptotic development

$$\theta(n) + \Theta(n) \{a_0 + a_1 n^{-1/p} + \cdots\}$$

for large values of n, if the function

$$(\chi(n) - \theta(n))/\Theta(n)$$

admits the asymptotic development

$$a_0 + a_1 n^{-1/p} + \cdots$$

in the sense defined above. Cp. Barnes, 1906, *Philosophical Transactions of the Royal Society of London*, Series A, Vol. 206, pp. 249-297.

2. The Lie of the Function $\Re f(z)$ on the Complex Plane.

2.1. We will write for convenience

$$R_f(z) = \Re f(z), \quad I_f(z) = \Im f(z)$$

and in general, when it is not necessary to make the reference to the function $f(z)$ explicit, these symbols may be replaced by the symbols $R(z)$, $J(z)$, simply. Now consider the surface consisting of the aggregate of points (x, y, R). By analogy with a relief map, in which at every point R represents the height of the land above sea-level, or some other standard of reference, we may call the lines on the surface on which R is constant the 'level' lines. On these lines J will be changing most rapidly and with the same analogy in view, these lines may be called the 'lines of greatest slope' or the 'lines of steepest descent or steepest ascent.' If at a point the partial differential coefficients of R and J with respect to x and y satisfy the equations

$$\partial R/\partial x = \partial J/\partial y, \quad \partial J/\partial x = -\partial R/\partial y$$

it follows that

$$\nabla^2 R = 0$$

and therefore R cannot at that point have a maximum or mimimum. At a point at which

$$d^s f/dz^s = 0 \quad (s = 1, 2, \cdots, p), \quad d^{p+1} f/dz^{p+1} \neq 0$$

there will be p lines of greatest slope and p level lines. We may say that a point z is a 'col' or 'pass' of order p, if at that point the first p differential coefficients of $f(z)$ with respect to z vanish and the $(p+1)th$ differential coefficient does not vanish.*

2.2. We are primarily concerned with the question of paths in the z-plane on which $R(z)$ has a certain upper bound. We will say that a point lies at the level M or on the M-level, if

$$R(z) = M.$$

We will call the aggregate of points which constitute the levels inferior to the level M, the 'underland M,' and the aggregate of points which constitute the levels superior to the M level, the 'overland M.' Thus the underland M is the class of points such that

$$R(z) = M - \delta$$

where δ is any positive number. Similarly the overland M is the class of points such that

$$R(z) = M + \delta$$

---

* The language of § 2.1 is taken from the papers cited on p. 4.

where δ is any positive number. Evidently with these definitions, every point in the z-plane belongs to the underland, overland or level associated with each number M. We will assume that the function $R(z)$ is a continuous function of x and y; then it follows that any path joining two points, one of which lies in the underland M and the other in the overland M, will traverse the level M at least once. The region which forms an underland or overland is plainly an open region. The limit of any sequence of points in an overland or underland either lies in the overland or underland in question, or lies on the level associated with the region. In fact, the region consisting of an underland M together with the level M is a closed region. An overland and its associated level, in the same way, form a closed region. With the assumption introduced above as to the continuity of $R(z)$, an underland evidently has one of the two

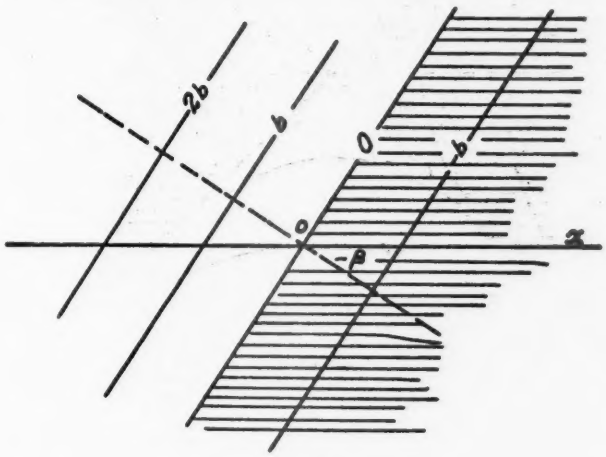


FIG. 2.31.

$f(z) = -B - Bz = -bre^{i\beta+i\theta}$.

o-underland shaded.

distinguishing marks of a continuum; for if a point z belongs to the underland M, points sufficiently near z will also belong to the underland M. The same is true of an overland. It remains to be discussed whether these regions possess the second characteristic of a continuum, namely, that between any two points belonging to one such region, there is a path lying wholly in the region in question.

2.3. We proceed to discuss the lie of the function $R_f(z)$ in various particular cases.

Let

$$f(z) = -Bz = -bre^{i(\beta+\theta)}. \tag{2.31}$$

The level line through the origin is $\sin(\beta + \theta) = 1$. This line divides the plane in the underland $M = 0$ and the overland $M = 0$. The underlands and overlands $M = b, M = 2b, \cdots, M = -b, M = -2b, \cdots$, abut along the levels $M = b, M = 2b, \cdots, M = b, M = -2b, \cdots$. Any two points belonging to the underland $M = 0$, may be joined by a path lying wholly in this region, and therefore, in this case, the underland $M = 0$ is a continuum. Similarly every other underland and every overland is a continuum.

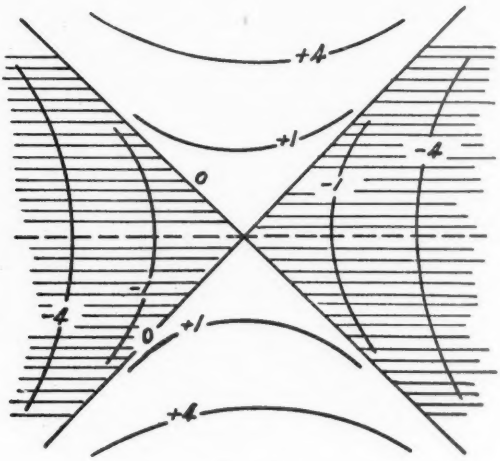


FIG. 2.32.

$f(z) = -z^2$.

o-underland shaded.

Let

$$f(z) = -z^2. \tag{2.32}$$

In this case there is a col of order two at the origin. Further, since

$$R = -x^2 + y^2; \quad J = -2xy,$$

the levels are hyperbolae with the lines

$$x^2 - y^2 = 0$$

as asymptotes, and the lines of greatest slope are hyperbolae with the lines

$$xy = 0$$

as asymptotes. The underland $M = 0$ is shaded in the diagram. Evidently, it is not the case that between any two points in this underland, a path can

be drawn lying entirely in the underland. To connect the two points $(a, 0)$, $(-a, 0)$ which lie in this underland, a path must enter the overland $M = 0$, or at least traverse the col at the origin. On the other hand, any two points which lie in one of the shaded quadrants can be connected by a path lying wholly in this quadrant. Each underland and each overland in fact forms two continua, in this particular case.

The distinction between an underland which forms one continuum and an underland which consists of more than one continuum is clearly essential for the purpose in hand. Thus take for example the function defined by the integral

$$\int e^{-nz^2} z^2 dz$$

taken along any path running from $(-a, 0)$ to $(a, 0)$. Both points $(a, 0)$, $(-a, 0)$ lie in the underland $M = 0$. But there exists no path between them, on which

$$R(-z^2) = y^2 - x^2 \geq -\delta$$

where δ is some positive number, and therefore it does not follow that

$$\int_{-a}^{+a} e^{-nz^2} z^2 dz \sim_n 0$$

If, however, we consider instead the integral taken along any path running from $(a, 0)$ to $(2a, 0)$, the situation is changed, for we may now select the straight line path from $(a, 0)$ to $(2a, 0)$, on the whole of which

$$R(-z^2) = y^2 - x^2 \leq -a^2$$

and we consequently deduce the result

$$\int_{a}^{2a} e^{-nz^2} z^2 dz \sim_n 0.$$

2.4. In order to draw attention to this important distinction between an underland which is such that any two points belonging to it may be joined by a path lying wholly in this underland, and an underland which is such that in the case of some pairs of points belonging to the underland there is no path joining them and lying wholly in the underland, we introduce the notion of a valley. We call a class of points an 'M-valley,' if the unattained upper bound of $R(z)$ for points in the class is M and if the points form a continuum. It

is occasionally convenient to refer to an M-valley as a valley whose index is M. In the same way, we might call a class of points an 'M-mountain continuum' if the unattained lower bound of $R(z)$ for the points in the class is M, and if the points form a continuum.

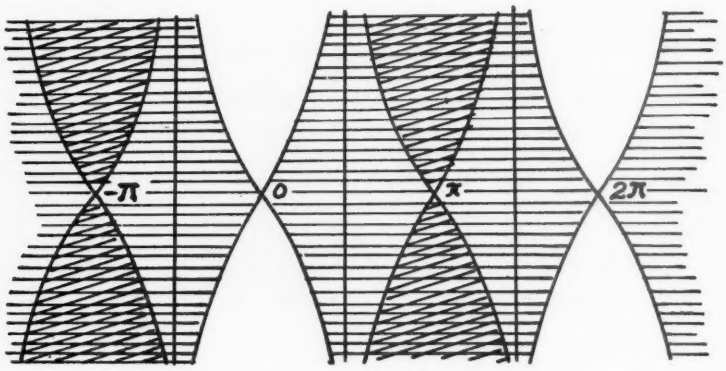


FIG. 2.41.

(+ 1)-underland

(— 1)-underland

$f(z) = \cos z$.

We may take as another illustration of the definitions the case when

$$f(z) = \cos z. \tag{2.41}$$

There are cols of order two at the points $(n\pi, 0)$. Those at the points $(2r\pi, 0)$ lie at the level + 1, those at the points $([2r + 1]\pi, 0)$, at the level — 1. The underland + 1, which is shaded in the figure, of course, contains the underland — 1, so that the lower cols together with part of the overland associated with the level — 1 on which they lie is contained in the underland + 1. The underland + 1 consists of an enumerable set of valleys. The question then arises as to the most convenient way of specifying any one valley of the enumerable set which in this case make up an underland. Of the valleys whose index is + 1, there are two and only two which have the origin on their boundaries; there are, again, two and only two of these valleys which have the point $(2r\pi, 0)$ on their boundaries. It is evidently a simple matter to specify one or other of the two valleys which have the point $(2r\pi, 0)$ on their boundaries, by citing

the directions of the lines of steepest descent through this point, which lie in the valleys. Indeed, we may conveniently call the two valleys which abut at the origin the $(\theta = 0)$ valley and the $(\theta = \pi)$ valley. The two valleys with index -1 which abut at the point $(\pi, 0)$ in the same way may be called the $(\theta = \pi/2)$ valley and the $(\theta = -\pi/2)$ valley.

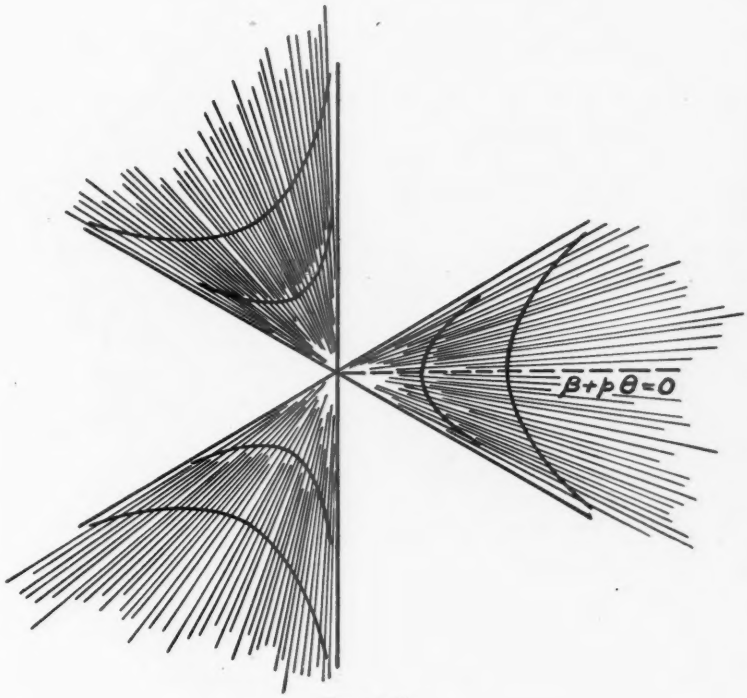


FIG. 2.42.

$f(z) = -Bz^p = -br^p e^{i\beta+ip\theta}$ $(p = 3)$.

o-underland shaded.

Thus to take another example, if

$$f(z) = -Bz^p = -br^p e^{i(\beta+p\theta)} \tag{2.42}$$

the level lines through the origin have the equation

$$\cos(\beta + p\theta) = 0$$

and the lines of greatest slope, the equation

$$\sin(\beta + p\theta) = 0.$$

The underland $M = 0$ falls into p valleys and the overland $M = 0$ into p mountain continua which all abut at the origin which is a col of order p. It will be convenient to specify any one of these valleys whose index is zero, by citing the direction of the line of steepest descent through the origin, which lies in the valley in question. In the diagram (which illustrates the case when $p = 3$), the valley $(\theta = -\beta/p)$ and the valley $[\theta = (\pm 2\pi - \beta)/p]$ are shaded. The remaining valleys are evidently the valleys

$$\theta = (4\pi - \beta)/p,\ (6\pi - \beta)/p, \cdots,\ [(2p-1)\pi - \beta]/p.$$

2.5. Suppose now that in the neighborhood of the point z_s, $f(z) - f(z_s)$ admits the convergent expansion

$$\begin{aligned} &-B(z-z_s)^p + \sum_{m=1}^{\infty} B_m (z-z_s)^{m+p} \\ &= -{}_sf_1(z) + \sum_{m=1}^{\infty} B_m (z-z_s)^{m+p} \\ &= -br e^{i\beta + ip\theta} + \sum_{m=1}^{\infty} b_m r^{m+p} e^{i\beta_m + ip\theta} \end{aligned}$$

where p is a positive integer and ${}_sb \neq 0$, and suppose further that

$$p\alpha_t + \beta = 2t\pi \qquad (t = 0, 1, \cdots, p-1).$$

Then, evidently at the point z_s, the function $f(z)$ has a col of order p. The lines through the point z_s whose equations are

$$\cos(\beta + p_s\theta) = 0, \quad \sin(\beta + p\theta) = 0$$

now represent the tangents to the level lines through the col and the tangents to the lines of greatest slope through the col, and in general not these lines themselves. Consider, however, the arrangement of points into valleys by the function

$${}_sf_1(z) = -B(z-z_s)^p.$$

The lines whose equation are

$$\cos(\beta + p\theta) = 1$$

which may also be written in the form

$$\theta = \alpha_t \qquad (t = 0, 1, \cdots, p-1)$$

are the lines of steepest descent themselves for the function ${}_sf_1(z)$ through the

col at the point z_s. Now the underland $M = 0$ for the function ${}_sf_1(z)$ is defined by the relation

$$\Re\, {}_sf_1(z) \leq -\delta < 0$$

and the underland $M = 0$ for the function $f(z) - f(z_s)$ by the relation

$$\Re[f(z) - f(z_s)] \leq -\delta < 0\,.$$

But

$$f(z) - f(z_s) = {}_sf_1(z) + \epsilon(z - z_s)^p$$

where ϵ can be made as small as we like, by taking $|\, z - z_s\,|$ sufficiently small. Thus if a positive number δ exists, such that

$$\Re[f(z) - f(z_s)] \leq -\delta < 0$$

it follows that a positive number δ' also exists, such that

$$\Re\, {}_sf_1(z) \leq -\delta' < 0.$$

In fact the points which form the underland $M = 0$ for the function ${}_sf_1(z)$ and are sufficiently near z_s also form the underland $M = 0$ for the function $f(z) - f(z_s)$. And further, the points sufficiently near the point z_s which form one valley for the one function also form one valley for the other function. Thus points sufficiently near the point z_s are segregated into valleys by the function ${}_sf_1(z)$ in the same way as they would be by the function $f(z) - f(z_s)$.*

2.6. We shall therefore adopt the following procedure in specifying valleys associated with the function $f(z)$. We shall say that the points (z_t) form a valley 'associated with the function $f(z)$ at the point z_s,' if the points form a valley and if there are points of this class in the neighbourhood of the point z_s. Further, we shall call a valley, 'the α_t-valley associated with the function $f(z)$ at the point z_s,' if it is the valley associated with the function $f(z)$ which contains as its line of greatest slope the line whose tangent at the point z_s is given by

$$z - z_s = re^{i\alpha_t}.$$

Thus if in the neighbourhood of the point z_s the function $f(z) - f(z_s)$ admits the convergent expansion

---

* It is plain that the fact that valleys are open regions, whatever the function with which they are associated, is the crux of this argument and that nothing of the kind is true with respect to the level lines through the points z_s, which are associated with the function ${}_sf_1(z)$ and the function $f(z) - f(z_s)$ respectively.

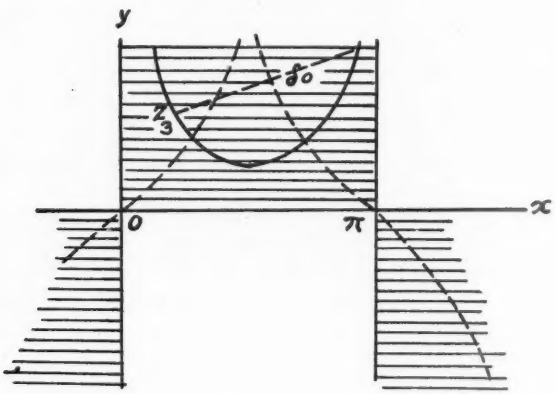
$$= -B(z - z_s)^p + \text{higher powers},$$
$$= -be^{i\beta}(z - z_s)^p + \text{higher powers},$$

there are evidently p valleys which abut at the point z_s, and with these definitions they will be called the valleys

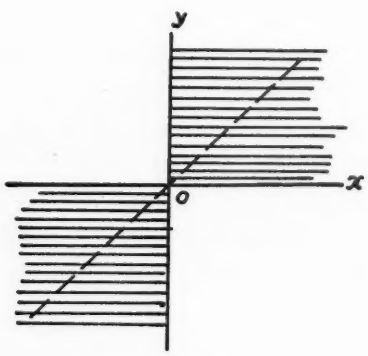
$$\theta = \alpha_t \qquad (t = 0, 1, \cdots, p-1).$$

where

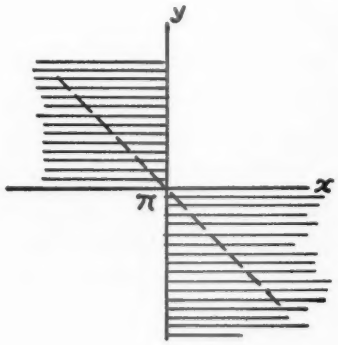
$$\beta + p\alpha_t = 2t\pi.$$



$f(z) - f(0) = -i(\cos z - 1)$: o-underland shaded.



$_0f_1(z) = iz^2/2!$.
o-underland shaded.



$_\pi f_1(z) = -(i/2!)(z-\pi)^2$.
o-underland shaded.

FIG. 2.61.

In particular, we may remark that associated with each valley of the function at the point z_s, there exists a positive number δ (which may be infinite) which depends on $f(z)$ and on z_s, and is such that the path

$$z - z_s = re^{i\alpha_t} \qquad 0 < r \leq \delta$$

consisting of a segment of the tangent to the line of steepest descent in the α_t-valley, lies wholly in this α_t-valley associated with the function $f(z)$ at the point z_s.

Consider the case of the function

$$f(z) = -i\cos z. \tag{2.61}$$

In the neighborhood of the origin, and indeed for all finite values of z,

$$f(z) = -i(1 - z^2/2! + \cdots).$$

Now compare the valleys associated with the function $f(z)$ at the origin and those associated with the function

$${}_0f_1(z) = iz^2/2! = r^2/2\ (\cos(\pi/2 + 2\theta) + i\sin(\pi/2 + 2\theta)).$$

Evidently the lines

$$z = re^{i\pi/4}, \quad z = re^{-3i\pi/4}$$

which are the lines of steepest descent for the function ${}_0f_1(z)$ at the origin lie within the valleys $(\theta = \pi/4)$ and $(\theta = -3\pi/4)$ associated with the function $f(z)$ at the origin so long as

$$r \leq \delta < \pi/2^{\frac{1}{2}}.$$

Again, the lines

$$z - \pi = re^{-i\pi/4}, \quad z - \pi = re^{3i\pi/4}$$

which are the lines of steepest descent for the function

$${}_\pi f_1(z) = -(i/2)(z - \pi)^2$$

at the point $(\pi, 0)$ lie in the valleys $(\theta = -\pi/4)$ and $(\theta = 3\pi/4)$ associated with the function $f(z)$ at the point $(\pi, 0)$, so long as

$$r \leq \delta < \pi/2^{\frac{1}{2}}.$$

All these considerations plainly apply also to the case of the valley associated with a point z_s, in the neighbourhood of which $f(z) - f(z_s)$ admits the convergent expansion

$$-B(z - z_s) + \sum_{m=1}^{\infty} B(z - z_s)^{m+1}.$$

Thus in the particular case under consideration, if

$$R(z_s) = M$$

$$\begin{aligned} f(z) - f(z) &= -i(\cos z - \cos z_s) \\ &= i\sin z_s (z - z_s) + \text{higher powers}, \end{aligned}$$

the tangent to the line of steepest descent at z_s, which is the normal to the level line at the point z_s, has equation

$$z - z_s = re^{ia}$$

where

$$ie^{ia} \sin z_s$$

is real and negative. The segment of this line given by

$$0 < r \geq \delta$$

will lie in the valley of $f(z)$ at this point, if $\delta < \delta_0$ where δ_0 is the length of the chord of the level line

$$R(z_s) = -M_s$$

through the point z_s.

NOTE.—The theory of underlands and valleys associated with a function of a complex variable may be developed step by step by the methods of Principia Mathematica. The primitive idea of the system is that of the class of points which form the M-level for the function $f(z)$. We will write

$$\text{lev}_f \text{`} M$$

for the M-level so that the symbol 'lev_f' denotes the relation between a class of points (z_s) and an ordinal number M, when the class (z_s) constitutes the level M for the function $f(z)$. The various levels associated with the various ordinal numbers may be ordered by the relation 'inferior to' which may be denoted by the symbol I. We shall say that one level is inferior to another level, when the ordinal number associated with the one level is less than the ordinal number associated with the other. Thus *

$$I = \text{lev}_f \text{`} <$$

To define the underland M we consider the levels inferior to the level M, and take the logical sum of all these classes. Thus the underland M, which may be written 'under 'M,' is given by

$$\text{under `} M = s \text{`} \vec{I} \text{`} \text{lev}_f \text{`} M.$$

Now consider k the class of levels inferior to the level M. Evidently

$$k = \vec{I} \text{`} \text{lev} \text{`} M.$$

Now k forms a segment of the relation I and the series of segments of the relation I, namely $s \text{`} I$, where †

$$s \text{`} I = I_{lc} \text{ t } D \text{`} I_c$$

is, plainly, ordinally similar to the series of underlands. The theory of the series of segments associated with various kinds of serial relations has been extensively discussed by Whitehead and Russell in Principia Mathematica,† 210-217. It is easily deducible from the theorems already known about series of segments that if a relation P is

* *Principia Mathematica*, 150.

† *Loc. cit.*, 170, 212.

Dedekindian, the relation P and the relation $s^{\cdot}P$ are ordinally similar. If therefore, we assume that the levels form a Dedekindian series, it at once follows that the underlands form an ordinally similar series. This fact is of importance in the question of the boundary of an underland. For, take the class of levels inferior to the M-level. Together with the M-level, the class forms a class which is closed * with respect to the relation I. We may then at once deduce that the underland M, which is an open region, forms with the level M a closed region. Other properties of underlands may be deduced from the definitions given above. In particular, properties of the various valleys into which underlands may be resolved may be deduced from assumptions as to the various simple curves into which levels may be resolved.

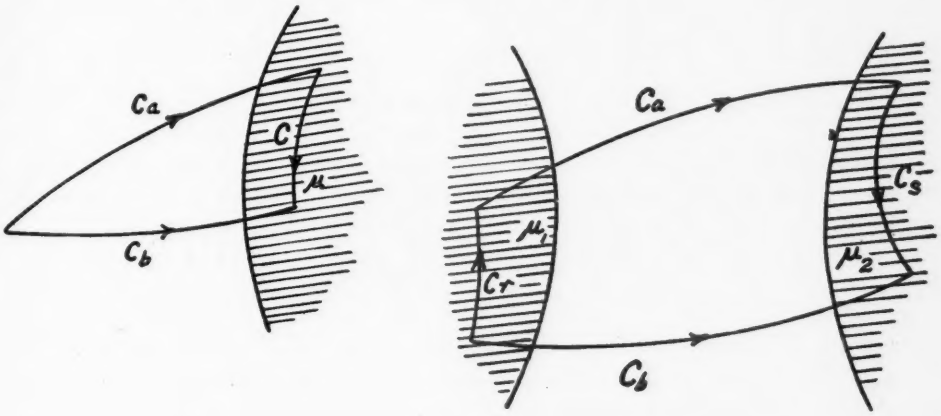


FIG. 3.1.

3. THE EXISTENCE THEOREMS.

3.1. The results of § 1.3 may now be stated in the following way:

1. If C_a and C_b are two paths starting at the same point and ending in m, an M-valley associated with the function $f(z)$,

$$e^{-nM}\int_{C_b} e^{nf(z)}F(z)\,dz \sim_n e^{-nM}\int_{C_a} e^{nf(z)}F(z)\,dz\,.$$

provided that there exists a third path C which lies in this valley and is such that the paths C_b and $C_a + C$ are equipollent for the integral

$$I = \int e^{nf(z)}F(z)\,dz\,.$$

2. If C_a and C_b are two paths starting in m_1, an M-valley and ending in m_2 a second M-valley associated with the function $f(z)$,

* *Loc. cit.*, 216.

$$e^{-nM}\int_{C_b} e^{nf(z)}F(z)\,dz \sim_n e^{-nM}\int_{C_a} e^{nf(z)}F(z)\,dz$$

provided that there exist paths C_r and C_s which lie in the first and second valleys respectively and are such that the paths C_b and $C_r + C_a + C_s$ are equipollent for the integral

$$I = \int e^{nf(z)}\,F(z)\,dz.$$

These theorems are existence theorems in the sense that they assert of a certain class of paths associated with certain M-valley continua that whichever paths of the class C_a and C_b may be

$$e^{-nM}\int_{C_b} e^{nf(z)}F(z)\,dz \sim_n e^{-nM}\int_{C_a} e^{nf(z)}F(z)\,dz$$

just as the first existence theorem of complex integration, Cauchy's theorem, asserts of a certain class of paths, that whichever paths of the class C_a, C_b may be

$$\int_{C_b} F(z)\,dz = \int_{C_a} F(z)\,dz.$$

On the one hand, Cauchy's theorem is restricted to classes of paths which start at one point z_1 and end at another point z_2, while the present theorem deals with classes of paths which end in an M-valley and start either at the one and the same point or in another M-valley. On the other hand the present theorem is restricted to asserting of any two paths C_a and C_b belonging to certain groups defined with reference to M-valleys, that

$$e^{-nM}\int_{C_b} e^{nf(z)}F(z)\,dz \sim_n e^{-nM}\int_{C_a} e^{nf(z)}F(z)\,dz$$

whereas Cauchy's theorem asserts of any two paths C_a and C_b belonging to certain groups that

$$\int_{C_b} F(z)\,dz = \int_{C_a} F(z)\,dz.$$

3.2. Many particular cases of these existence theorems occur in the theory of asymptotic expansions. To take the simplest case of all, consider the integral

$$I(n) = \int e^{-Nz}\,z^{k-1}\,dz = \int e^{-nze^{i\nu}}\,z^{k-1}\,dz$$

taken along various paths in the z-plane. Let

$$\Re\,(N) > 0, \quad k > 0.$$

If C be any path starting at the origin and ending anywhere in the C-valley associated with the function

(3.21) $$f(z) = -ze^{i\nu}$$

(the half plane shaded in the figure)

$$\int e^{-nze^{i\nu}} z^{k-1}\, dz \sim_n \int e^{-nze^{i\nu}} z^{k-1}\, dz = [\Gamma(k)/n^k]e^{-i\nu k}.$$

This fact is evidently the reason for the importance of the Gamma function

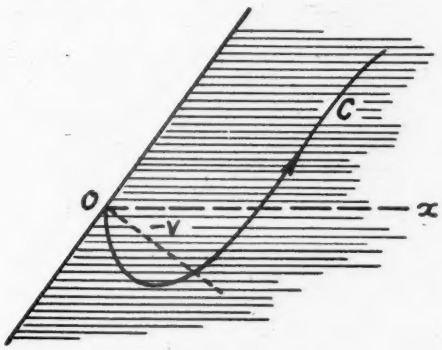


FIG. 3.21.
$f(z) = -ze^{i\nu}$.

in the theory of asymptotic expansions. Whatever the path C may be, it is in general a simple matter to relate the function

$$I_C(n) = \int_C e^{-nze^{i\nu}} F(z)\, dz$$

to a set of integrals which may be expressed in terms of Gamma functions.*

3.3. The proviso in the existence theorems as to the equipollence of certain paths for the integral I raises an interesting point. For suppose

(3.31) $$I = \int [e^{-nz}/(z-a)]\, dz,$$

and consider the integral taken along the set of paths which start at the origin and end at points in the half plane

$$x > 0$$

* Cp. Barnes, *loc. cit.*, p. 254. Watson, *loc. cit.*, *Proceedings of the London Mathematical Society*, p. 133.

which is the underland 0 for the function

$$f(z) = -z$$

and indeed the one and only 0-valley for this function.

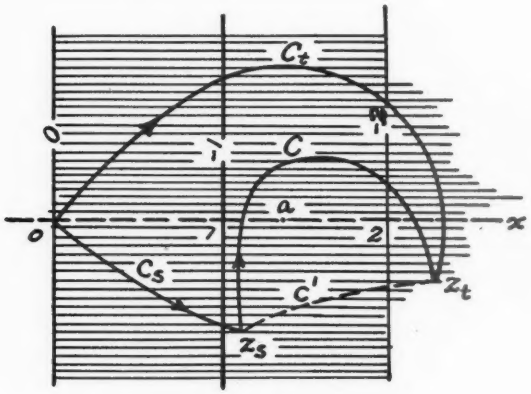


FIG. 3.31.
$f(z) = -z$.

Two paths C_s and C_t ending at the points z_s and z_t are shewn in the diagram. In spite of the existence of the pole of the function

$$F(z) = 1/(z-a)$$

at the point $(a, 0)$, there exists a path C from z_s to z_t which is such that the

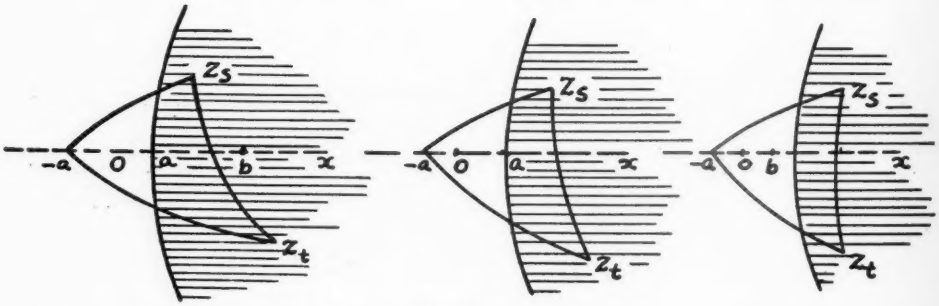


FIG. 3.32. $f(z) = -z^2$.

$b > a$ $\quad$ $b = a$ $\quad$ $b < a$

paths C_t and $C_s + C$ are equipollent for the integral I. Therefore evidently

$$\int_{C_s} [e^{-nz}/(z-a)]\, dz \sim_n \int_{C_t} [e^{-nz}/(z-a)]\, dz.$$

Now the residue of the integrand at the pole $z = a$, is e^{-na}, so that if a path C' had been selected, which with the paths C_s and C_t enclosed this point, we should have had the result

$$\int_{C_s} [e^{-nz}/(z-a)]\,dz - \int_{C_t} [e^{-nz}/(z-a)]\,dz = 2\pi i\, e^{-na}$$

so that as before

$$\int_{C_s} [e^{-nz}/(z-a)]\,dz \sim_n \int_{C_t} [e^{-nz}/(z-a)]\,dz.$$

On the other hand consider the integral

$$\int [e^{-nz^2}/(z-b)]\,dz \tag{3.32}$$

along a set of paths starting at the point $(a, 0)$ and ending an M-valley where

$$M = \mathfrak{R}(-z^2)_{z=a} = -a^2.$$

We will select as the M-valley to be considered the continuum which contains the portion of the real axis on which x is positive and sufficiently large. The boundary of this valley is evidently one of the branches of the hyperbola

$$x^2 - y^2 = a^2.$$

Evidently, if $b = b_1 > a$, the pole of the integrand lies in the M-valley in question. We therefore have as in the previous case

$$e^{-nM} \int_{C_s} [e^{-nz^2}/(z-b_1)]\,dz \sim_n e^{-nM} \int_{C_t} [e^{-nz^2}/(z-b_1)]\,dz$$

whichever paths of the set of paths starting at $(a, 0)$ and ending in the M-valley, C_s and C_t may be. If however $b = b_2 \leqq a$, this result does not persist, and in particular we do not have the result that

$$\int_{C_s} [e^{-nz^2}/(z-b_2)]\,dz + \int_{C} [e^{-nz^2}/(z-b_2)]\,dz = \int_{C_t} [e^{-nz^2}/(z-b_2)]\,dz$$

in the case of the paths shewn in the diagram. This conclusion would carry the consequence that

$$e^{na^2} \int_{C_s} [e^{-nz^2}/(z-b_2)]\,dz - e^{na^2} \int_{C_t} [e^{-nz^2}/(z-b_2)]\,dz = O(e^{-n\delta})$$

where δ is some positive number, whereas in fact

$$\left(\int_{C_s} + \int_{C}\right) [e^{-n(z^2-a^2)}/(z-b_2)]\,dz - \int_{C_t} [e^{-n(z^2-a^2)}/(z-b_2)]\,dz = 2\pi i\, e^{n(a^2-b_2^2)}.$$

and

$$\int [e^{-n(z^2-a^2)}/(z-b_2)]\,dz = O(e^{-n\delta}) \qquad (\delta > 0)$$

so that

$$\left(\int_{C_s} - \int_{C_t}\right) [e^{-n(z^2-a^2)}/(z-b_2)]\,dz \sim_n 2\pi i\, e^{n(a^2-b_2^2)}.$$

4. THE STANDARD PATHS ASSOCIATED WITH THE FUNCTION $f(z)$.

4.1. In the theory of a function defined by the integral

$$I = \int F(z)\,dz$$

taken along a path C, the first problem is the specification of a class of paths which is such that the integral along all these paths is the same. Cauchy's theorem is concerned with this problem. The next problem is the exaluation of the function

$$I_C = \int_C F(z)\,dz$$

for as wide a class of paths as possible. Suppose that C_s is the path consisting of the circular arc

$$z - z_s = \rho e^{i\theta} \qquad -\pi \leq \theta \leq \pi$$

and evaluate first

$$\int_{C_s} (z-z_s)^{A-1}\,dz$$

where the integrand is the one-valued function

$$(z-z_s)^{A-1} = e^{(A-1)\log(z-z_s)}$$

and $\log(z-z_s)$ is purely real on the line on which $(z-z_s)$ is real and positive. We easily obtain the result that

$$\int_{C_s} (z-z_s)^{A-1}\,dz = 2i\rho^A \sin \pi A/A.$$

If now we consider the integral

$$I = \int F(z)\,dz$$

along the same path C_s, when, in the neighbourhood of the point z_s, $F(z)$ admits the convergent development

$$\sum_{m=0}^{\infty} A_m (z-z_s)^{A+m-1},$$

for sufficiently small values of ρ we may integrate the series for $F(z)$ term-by-term and we obtain the result

$$I_{C_s} = 2i\{(\sin \pi A/A)\rho^A + [\sin \pi(A+1)/(A+1)]\rho^{A+1} + \cdots\}$$

the series on the right being evidently convergent. This result enables us to evaluate a function defined by the integral I taken along any path C which is equipollent for the integral I to a standard path consisting of the circular arc

$$z - z_s = \rho e^{i\theta} \qquad -\pi \leq \theta \leq \pi$$

provided that in the neighbourhood of the point z_s, $F(z)$ admits the convergent expansion

$$\sum_{m=0}^{\infty} A_m (z - z_s)^{A+m-1}$$

and that ρ is sufficiently small. If $A = -p$, where p is a positive integer or zero,

$$I_{C_s} = 2iA_p \operatorname*{lt}_{A+p \to 0} [\sin \pi(A+p)/(A+p)]\rho^{A+p} = 2\pi i A_p$$

and we obtain the theorem of residues, which is indeed a particular case of integration along the standard path C_s.

4.2. The procedure in the theory of the asymptotic evaluation of a function defined by an integral taken along a certain path is entirely similar. We take two types of standard paths.

Suppose z_s is a point in the neighbourhood of which $f(z) - f(z_s)$ admits the convergent expansion

$$-B(z - z_s)^p + \sum_{m=0}^{\infty} B_m (z - z_s)^{p+m}$$

where p is a positive integer. Let $\theta = a$ be one of the p valleys associated with the function $f(z)$ at the point, and consider the straight line path

$$z - z_s = re^{ia} \qquad 0 \leq r \leq \delta$$

We will choose δ sufficiently small to ensure that this path shall end at a point in this a-valley. It will be convenient to denote this path by the symbol

$$z_s(\alpha)_\delta.$$

If now C is a path which starts at the point z_s and ends in this a-valley which

is a valley associated with the function at the point z_s, then it follows, in general, that *

$$\int_C e^{nf(z)-nf(z_s)} F(z)\, dz \sim_n \int_{z_s(\alpha)\delta} e^{nf(z)-nf(z_s)} F(z)\, dz.$$

We will call a path which starts at a point z_s and ends in a valley associated with $f(z)$ *at that point,* 'a path of the first type'; in particular, we will call the path $z_s(\alpha)_\delta$ which is itself a path of the first type, 'a standard path of the first type.' Then evidently, the asymptotic evaluation of the integral

$$\int e^{nf(z)-nf(z_s)}\ F(z)\, dz$$

along any path of the first type starting at z_s can in general be derived from the asymptotic evaluation of this integral along the appropriate standard path of the first type.

4.3. The second existence theorem suggests a second type of standard path. Suppose that z_s is a point in the neighbourhood of which the function $f(z)$ admits the convergent expansion

$$-B(z-z_s)^p + \sum_{m=1}^{\infty} B_m (z-z_s)^{p+m}$$

where p is a positive integer greater than unity. Then there will be at least two valleys associated with the function $f(z)$ which abut at this point. Of these valleys, let

$$\theta = \alpha_1, \qquad \theta = \alpha_2$$

be two. Consider now the path consisting of the circular arc

$$z - z_s = \delta e^{i\theta} \qquad\qquad \alpha_1 \leq \theta \leq \alpha_2$$

We will choose δ sufficiently small to ensure that this path shall begin in the α_1-valley associated with the function $f(z)$ at the point z_s and shall end in the α_2-valley associated with the function $f(z)$ at the same point. It will be convenient to denote this path by the symbol

$$z_s(\alpha_1, \alpha_2)\delta$$

---

* It is evidently necessary to assume that there exists a path C_a which lies in this valley $\theta = a$ and is such that the paths $C + C_a$ and $z_0(a)_\delta$ are equipollent for the integral in question. Cp. § 3.1.

If now C is a path starting in this α_1-valley and ending in this α_2-valley (both the valleys being associated with the function $f(z)$ at one point), it follows, in general,* that

$$\int_C e^{nf(z)-nf(z_s)} F(z)\, dz \sim_n \int_{z_s(\alpha_1,\alpha_2)\delta} e^{f(z)-nf(z_s)} F(z)\, dz.$$

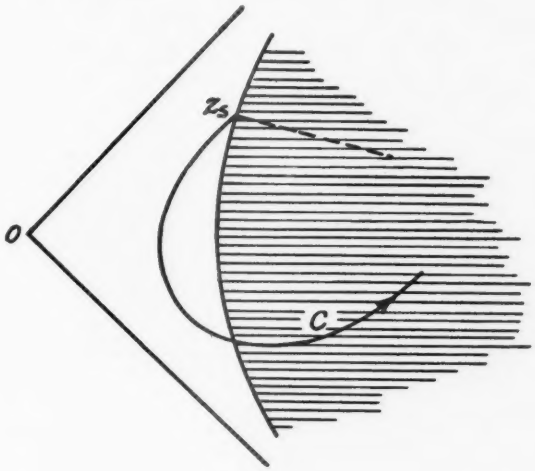


FIG. 4.41.

$f(z) = -z^2$.

We will call a path which starts in one valley associated with the function $f(z)$ at a point z_s and ends in a second valley associated with the function $f(z)$ at the same point, 'a path of the second type'; in particular, we will call the path $z_s(\alpha_1, \alpha_2)_\delta$ which is itself a path of the second type, 'a standard path of the second type.' Then evidently the asymptotic evaluation of the integral

$$\int e^{nf(z)-nf(z_s)}\ F(z)\, dz$$

along any path of the second type associated with the point z_s can, in general, be derived from the asymptotic evaluation of this integral along the appropriate standard path of the second type.

4.4. The diagram shows the standard path of the first type associated with the function

(4.41) $$f(z) = -z^2$$

* It is evidently necessary to assume that there exist paths C_{α_1} and C_{α_2} which lie in the valleys α_1 and α_2 respectively, and which are such that the paths $C_{\alpha_1} + C + C_{\alpha_2}$ and $z_s(\alpha_1, \alpha_2)_\delta$ are equipollent for the integral in question. Cp. § 3.1.

at a point z_s. Evidently the asymptotic evaluation of the integral

$$\int e^{nz^2-nz_s^2} F(z)\, dz$$

along this standard path would yield the asymptotic evaluation of the integral along any path C whatever which starts at the point z_s and ends within the section of the plane bounded by the branch of the hyperbola which passes through the point z_s.

The diagram shows a standard path of the second type associated with the function

(4.42) $$f(z) = i(z - \sin z)$$

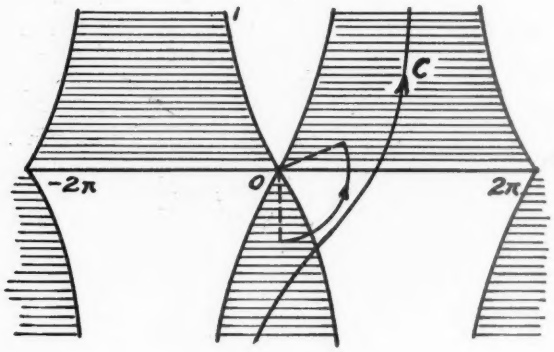


FIG. 4.42.
$f(z) = i(z - \sin z)$.
o-underland shaded.

at the origin. If C is any path running from $(a - i\infty)$ to $(b + i\infty)$ where

$$-\pi/2 < a < \pi/2, \qquad \pi/2 < b < 3\pi/2$$

evidently, in general,

$$\int_C e^{in(z-\sin z)} F(z)\, dz \sim_n \int_{O(-\pi/2,\, \pi/6)\delta} e^{in(z-\sin z)} F(z)\, dz.$$

5. THE ASYMPTOTIC EVALUATION OF THE INTEGRAL ALONG THE STANDARD PATHS.

5.1. In evaluating the integral

$$\int e^{nf(z)-nf(z_s)} F(z)\, dz$$

along the standard paths of the first and second type, we may evidently

restrict ourselves to the case when the point z_s is at the origin without any loss in generality.

In the course of the discussion of the integral

$$I = \int e^{nf(z)} z^{A-1} \phi(z)\, dz$$

along a certain class of paths, Perron * has obtained the asymptotic value of the integral I taken along a straight line path $O(\alpha)_\delta$. If we assume that the real part of A is positive and that in the neighbourhood of the origin the functions $f(z)$ and $\phi(z)$ admit the convergent developments

$$f(z) - f(0) = -Bz^p + \sum_{m=1}^{\infty} B_m z^{p+m} = -br^p e^{i\beta + ip\theta} + \sum_{m=1}^{\infty} B_m z^{p+m}$$

$$\phi(z) = \sum_{m=0}^{\infty} A_m z^m$$

where p is a positive integer and

$$p\alpha_t + \beta = 2t\pi$$

it follows, from Perron's analysis, that for sufficiently small values of δ,

$$\int_{O(\alpha_t)\delta} e^{nf(z)-nf(0)} z^{A-1} \phi(z)\, dz \sim 1/p \sum_{k=0}^{\infty} P_k \Gamma(A+k/p)\, e^{i\alpha_t(A+k)} (n^{1/p} b)^{-(A+k)}$$

where P_k is the coefficient of z^k in the expansion in powers of z of the function

$$\sum_{m=0}^{\infty} A_m z^m \times [1 - (B_1 z + B_2 z^2 \cdots)/B_0]^{-(A+k)/p}.\dagger$$

5.2. This result may be extended. Suppose that in the neighbourhood of the origin, $\phi(z)$ admits the *asymptotic* development

$$A_0 + A_1 z + A_2 z^2 + \cdots.$$

for values of arg z between $\alpha_t \pm \gamma$ where $\gamma \geq \gamma_0 > 0$.

Let

$$\phi_k(z) = \sum_{m=0}^{k-1} A_m z^m.$$

Then the condition satisfied by $\phi(z)$ in this case implies that

$$| \phi(z) - \phi_k(z) | \leq K | z |^k$$

---

* *Loc. cit.*

† A theorem which is substantially the particular case of this theorem which results when

$$f(z) = -Bz^p$$

was proved independently by Watson, 1916, *Proceedings of the London Mathematical Society*, Series 2, 17, p. 133. See also *Bessel Functions* by the same author, § 8.3.

for small values of z. Thus, if δ be chosen sufficiently small,

$$\left| \int_{O(a_1)\delta} e^{nf(z)-nf(0)} z^{A-1} (\phi(z) - \phi_k(z))\, dz \right| \leq K \left| \int_{O(a_1)\delta} e^{nf(z)-nf(0)} z^{A+k-1}\, dz \right|$$

and by the foregoing result

$$\int_{O(a_1)\delta} e^{nf(z)-nf(0)} z^{A+k-1}\, dz = 1/p\ \Gamma\left(\frac{A+k}{p}\right) \frac{e^{ia_1(A+k)}}{(n^{1/p}b)^{A+k}} (1 + O(1/n)^{1/p})$$

Thus

$$\int_{O(a_1)\delta} e^{nf(z)-nf(0)} z^{A-1} \phi(z)\, dz = \int_{O(a_1)\delta} e^{nf(z)-nf(0)} z^{A-1} \phi_k(z) + O(1/n)^{[\Re(A)+k]/p}.$$

But consider now the integral

$$I_k = \int_{O(a_1)\delta} e^{nf(z)-nf(0)A-1}\ \phi_k(z)\, dz.$$

The function $\phi_k(z)$ in the integrand is merely a polynomial in z of order $k-1$. The original theorem therefore again applies and we may at once assert that for sufficiently small values of δ,

$$I_k \sim 1/p \sum_{\nu=0}^{\infty} \Gamma(A + \nu/p)\ {}_kP_\nu\ e^{ia_1(A+\nu)}\ (n^{1/p}b)^{-(A+\nu)}$$

where ${}_kP_\nu$ is the coefficient of z^ν in the expansion in powers of z of the function

$$(A_0 + A_1 z + \cdots + A_{k-1} z^{k-1}) \times [1 - (B_1 z + B_2 z^2 + \cdots)/B]^{-(A+\nu)/p}.$$

With this definition of the ${}_kP_\nu$-coefficients, we have for sufficiently small values of δ the result that

$$\int_{O(a_1)\delta} e^{nf(z)-nf(z0)} z^{A-1} \phi(z) = 1/p \sum_{\nu=0}^{k-1} \Gamma\left(\frac{A+\nu}{p}\right) {}_kP_\nu\ \frac{e^{ia_1(A+\nu)}}{(n^{1/p}b)^{A+\nu}} + O(1/n)^{[\Re(A)+k]/p}.$$

Now when $\nu \leq k-1$ evidently

$${}_kP_\nu = P_\nu$$

where P_ν is the coefficient of z^ν in the expansion in powers of z of the function

$$\sum_{m=0}^{\infty} A_m z^m \times [1 - (B_1 z + B_2 z^2 + \cdots)/B]^{-(A+\nu)/p}.$$

We may therefore write the result in the more convenient form: if δ be sufficiently small

$$\int_{O(a_1)\delta} e^{nf(z)-nf(0)} z^{A-1} \phi(z)\, dz \sim 1/p \sum_{k=0}^{\infty} \Gamma\left(\frac{A+k}{p}\right) P_k\ \frac{e^{ia_1(A+k)}}{(n^{1/p}b)^{A+k}}.$$

The result of §5.1 is therefore true even if $\phi(z)$ satisfies the milder condition of having an asymptotic development in the neighbourhood of the origin which is valid at least for a small sector enclosing the straight line path of integration.

5.3. Perron also obtains the asymptotic value of the integral

$$\int e^{nf(z)-nf(0)}\, z^{A-1}\, \phi(z)\, dz$$

along the standard path of the second type $O(\alpha_1, \alpha_2)\delta$. If * we assume that in the neighbourhood of the origin, the functions $f(z)$ and $\phi(z)$ admit the same convergent developments as before, it follows that, if δ be chosen sufficiently small

$$\int_{O(\alpha_1,\alpha_2)\delta} e^{nf(z)-nf(0)}\, z^{A-1}\, \phi(z)\, dz \sim 1/p \sum_{k=0}^{\infty} \Gamma\left(\frac{A+k}{p}\right) P_k \frac{e^{i\alpha_2(A+k)} - e^{i\alpha_1(A+k)}}{(n^{1/p}b)^{A+k}}$$

where P_k is the coefficient of z^k in the expansion in powers of z of the function

$$\sum_{m=0}^{\infty} A_m z^m \times [1-(B_1 z + B_2 z^2 + \cdots)/B]^{-(A+k)/p}.$$

5.4. It is evidently possible to extend this result as in the case of the previous result. If instead of having a convergent development in the neighbourhood of the origin, $\phi(z)$ admits the asymptotic development

$$A_0 + A_1 z + \cdots$$

for values of arg z between γ_1 and γ_2 where

$$\gamma_1 < \alpha_1 < \alpha_2 < \gamma_2,$$

similar arguments to those used above will enable us to prove that it is still the case that

$$\int_{O(\alpha_1,\alpha_2)\delta} e^{nf(z)-nf(0)}\, z^{A-1}\phi(z)\, dz \sim 1/p \sum_{k=0}^{\infty} \Gamma\left(\frac{A+k}{p}\right) P_k \frac{e^{i\alpha_2(A+k)} - e^{i\alpha_1(A+k)}}{(n^{1/p}b)^{A+k}}$$

---

* A theorem which is substantially this theorem in the particular case when

$$f(z) = -Bz$$

was proved by Barnes, *loc. cit.*, p. 254.

where P_k is the coefficient of z^k in the expansion in powers of z of the function

$$\sum_{m=0}^{\infty} A_m z^m \times [1-(B_1 z + B_2 z^2 + \cdots)/B]^{-(A+k)/p}$$

provided only that δ be chosen sufficiently small.

5.5. The asymptotic evaluation of the integral

$$\int e^{nf(z)-nf(0)} F(z)\, dz$$

taken along standard paths of the first and second types associated with the origin has now been carried out for the case when $f(z)$ and $F(z)$ satisfy certain conditions. It is therefore possible to write down the asymptotic value of the integral

$$\int e^{nf(z)-nf(0)} F(z)\, dz$$

taken along any path of the first or second type associated with the origin, when $f(z)$ and $F(z)$ satisfy certain conditions in the neighbourhood of the origin and by a change of origin to write down the asymptotic value of the integral

$$\int e^{nf(z)-nf(z_s)} F(z)\, dz$$

taken along any path of the first or second types associated with any point z_s, given that the functions $f(z)$ and $F(z)$ satisfy corresponding conditions in the neighbourhood of the point z_s.*

---

* Certain writers (*loc. cit.*, p. 4) using the so-called method of steepest descents (Méthode du Col, Methode der Sattelpunkte), have discussed the integral

$$\int e^{nf(z)-nf(z_s)} F(z)\, dz$$

along a path C which passes through a point z_s at which $f'(z)$ vanishes and has the equation

$$I f(z) = I f(z_s)$$

when $f(z)$ and $F(z)$ are various special functions. (Cp. Watson, "Bessel Functions," § 8.3 for an account of the method.) This procedure of integrating along the line of steepest descent through a point at which two or more valleys abut, is, of course, suggested by the fundamental idea of the relation between the behaviour of the real part of $f(z)$ on a path and the asymptotic value for large values of n, of the integral

$$\int e^{nf(z)} F(z)\, dz$$

along that path. It is, however, more convenient to adopt as standard paths for the asymptotic evaluation of functions defined by the integral I along various pathe associated with a point z_s, not the line of steepest descent through z_s, but the small segments of the tangents to the lines of steepest descent through the point, and the circular arcs of small radius bounded by the tangents to lines of steepest descent lying in different valleys associated with the function $f(z)$ at the point z_s. For, we then require infor-

5.6. An interesting fact will be observed in connexion with the asymptotic evaluation of the integral

$$\chi(n) = \int_{O(a_1,a_2)\delta} e^{nf(z)-nf(0)}\,\phi(z)\,z^{A-1}\,dz.$$

It has been proved that, whether the real part of A is positive or not,

$$\chi(n) \sim 1/p \sum_{k=0}^{\infty} P_k\,\Gamma\left(\frac{A+k}{p}\right)\frac{e^{ia_2(A+k)} - e^{ia_1(A+k)}}{(n^{1/p}b)^{A+k}},$$

so that whether the real part of A is positive or not

$$\int_{O(a_1,a_2)\delta} e^{nf(z)}\,\phi(z)\,z^{A-1}\,dz = Ke^{nf(0)}\,n^{-A/p}\,\{1 + O(1/n^{1/p})\}.$$

---

mation as to the behaviour of $f(z)$ and $F(z)$ in the neighbourhood of the point z_s, simply, instead of requiring information as to their behaviour all the way along the line of steepest descent in question. Further, there are cases (as given above) when the integral along the lines of steepest descent from one valley associated with the function at the point z_s into a second valley associated with the function as that point does not exist, owing to an infinity of $F(z)$ at the point z_s, when the integral along the standard path of the second type consisting of a circular arc about the point z_s, exists. The notion of integrating along lines of steepest descent is therefore not essential to the development of the idea of the relation between the behaviour of the real part of the function $f(z)$ on a path C and the asymptotic value for large values of n of the integral I along that path. No prominence is given, therefore, to the name 'method of steepest descents.'

We are more concerned, however, to emphasize the fact that nothing whatever depends upon the point z_s being a point at which $f'(z)$ vanishes. (This fact is implicit in the papers of Brillouin and Perron already cited.) The functions defined by the integral

$$\int e^{nf(z)-nf(z_s)}\;F(z)\,dz$$

along paths from z_s which end in a valley associated with the function $f(z)$ at the point z_s, admit the same asymptotic development for large values of n, whether z_s is a point at which $f'(z)$ vanishes or not. In the latter case there will be only one valley associated with the function $f(z)$ at the point z_s instead of several. We adopt as a standard path the small segment of the tangent to the line of steepest descent through the point z_s. Analytically this case presents no special features with respect to the asymptotic evaluation of the integral

$$\int e^{nf(z)-nf(z_s)}\;F(z)\,dz$$

along paths which start at z_s and end in the α-valley (the one and only valley) associated with the function $f(z)$ at the point z_s. We have in fact that

$$\int e^{nf(z)-nf(z_s)}\;(z-z_s)^{A-1}\phi(z)\,dz$$

along such a path is, in general, asymptotically equivalent to

$$\sum_{\nu=0}^{\infty}\Gamma\left(\frac{A+\nu}{p}\right)P_\nu\;\frac{e^{ia(A+\nu)}}{(n^{1/p}b)^{A+\nu}}$$

and $p = 1$. With points at which two, three, . . . valleys abut, p has other special values. For this reason, no prominence is given to the names 'Méthode du col,' 'Methode der Sattelpunkte.'

where k is independent of n. Now if there is a path running from δe^{ia_1} to δe^{ia_2} on which the upper bound of the real part of $f(z)$ is M, it follows that

$$\int_{O(a_1,a_2)\delta} e^{nf(z)}\,\phi(z)\,z^{A-1}\,dz = O(e^{nM}).$$

When the real part of A is positive, we may choose as a path running from δe^{ia_1} to δe^{ia_2}, the rays

$$z = re^{ia_1}, \quad z = re^{ia_2} \qquad 0 \leq r \leq \delta$$

and on these rays the upper bound of the real part of $f(z)$ is evidently equal to the value of the real part of $f(z)$ at the origin, and we have the result

$$\int_{O(a_1,a_2)\delta} e^{nf(z)}\,\phi(z)\,z^{A-1}\,dz = O(e^{n\Re f(0)}),$$

a result which agrees with the statement

$$\int_{O(a_1,a_2)\delta} e^{nf(z)}\,\phi(z)\,z^{A-1}\,dz = Ke^{nf(0)}\,n^{-A/p}\,\{1 + O(1/n^{1/p})\}$$

in the case in question, namely when the real part of A is positive.

However, when the real part of A is no longer positive, the fact that the statements

$$\int_{O(a_1,a_2)\delta} e^{nf(z)}\,z^{A-1}\,\phi(z)\,dz = O(e^{n\Re f(0)}),$$

$$\int_{O(a_1,a_2)\delta} e^{nf(z)}\,z^{A-1}\,\phi(z)\,dz = Ke^{nf(0)}\,n^{-A/p}\,\{1 + O(1/n^{1/p})\}$$

are no longer in accord, is reflected in the lie of the function on the plane of z. For since the real part of A is no longer positive, any path from δe^{ia_1} to δe^{ia_2} must avoid the origin. But any path which starts in one valley associated with the function $f(z)$ at the origin and ends in another valley associated with the function $f(z)$ must either cross the origin at least once—which is impossible in this case though perfectly possible in the previous case when the real part of A was positive—or pass some distance over a mountain region associated with the function $f(z)$ at the origin. Thus on any path from δe^{ia_1} to δe^{ia_2} the upper bound of the real part of $f(z)$ is greater than the real part of $f(z)$ at the origin. However small η may be there exists a path from δe^{ia_1} to δe^{ia_2} on which the upper bound of the real part of $f(z)$ is equal to

$$\Re f(0) + \eta.$$

Thus we may assert that

$$\int_{\delta e^{ia_1}}^{\delta e^{ia_2}} e^{nf(z)}\,F(z)\,dz = O(e^{n[\Re f(0)+\eta]})$$

where η is any positive number however small. But we cannot assert the limit of this set of assertions, namely that

$$\int_{\delta e^{ia_1}}^{\delta e^{ia_2}} e^{nf(z)} F(z)\, dz = O(e^{n\Re f(0)}).$$

The order of magnitude of the integral is, in fact, given by the result

$$\int_{\delta e^{ia_1}}^{\delta e^{ia_2}} e^{nf(z)} F(z)\, dz = 1/p\ e^{nf(0)}\, \Gamma\left(\frac{A}{p}\right) \frac{e^{ia_2A} - e^{ia_1A}}{(n^{1/p}b)^A} [1 + O(1/n)^{1/p}]$$

a result which is in complete accord with the fact that whatever the value of η

$$\int_{\delta e^{ia_1}}^{\delta e^{ia_2}} e^{nf(z)} F(z)\, dz = O(e^{n(\Re f(0)+\eta)})$$

and that it is not the case that

$$\int_{\delta e^{ia_1}}^{\delta e^{ia_2}} e^{nf(z)} F(z)\, dz = O(e^{n\Re f(0)}).$$

6.1. It has proved possible to obtain the asymptotic evaluation of the integral

$$\int_C e^{nf(z)-nf(z_s)} F(z)\, dz$$

when C is a path of the first or second type associated with a point z_s, under certain conditions.

Suppose now that C is a path which can be subdivided into a finite or enumerable set of paths of the first or second type associated with a sequence of points (z_t). If C_t be the path associated with the point z_t, then

$$\int_{C_t} e^{nf(z)-nf(z_t)} F(z)\, dz \sim_n \int_{\gamma_t} e^{nf(z)-nf(z_t)} F(z)\, dz$$

where γ_t is the appropriate standard path. Now suppose that of the various points (z_t) with which the paths are associated, the subset (z_τ) are at the same level M, which is superior to the levels of the points of the set not belonging to the subset. Then, evidently

$$e^{-nM} \int_{C_t} e^{nf(z)} F(z)\, dz \sim_n 0$$

except in the case of the paths (C_τ) associated with the subset (z_τ). Therefore

$$e^{-nM}\int_C e^{nf(z)}F(z)\,dz \sim_n e^{-nM}\sum{}'\int_{C_\tau} e^{nf(z)}F(z)\,dz$$
$$\sim_n e^{-nM}\sum{}'\int_{\gamma_\tau} e^{nf(z)}F(z)\,dz$$

the prime denoting that the summation is taken only over the subset of paths associated with the subset (z_τ). The asymptotic evaluation is therefore complete.

6.2. It is of interest to consider the case of the asymptotic evaluation is therefore the integral

$$\int_{-\pi}^{\infty} e^{2n\cos z}\,e^{-z^3}\,dz$$

for the integration has to be carried out over an enumerable infinity of cols. We subdivide the path into the enumerable infinity of paths

$$(-\pi,\pi),\quad (\pi,3\pi),\cdots,\quad ([2r-1]\pi,\ [2r+1]\pi),\cdots$$

and evidently the path from $-\pi$ to $+\pi$ is a path of the second type associated with the origin.* Thus

$$\int_{-\pi}^{+\pi} e^{2n(\cos z-1)}\,e^{-z^3}\,dz \sim \tfrac{1}{2}\sum_{\nu=0}^{\infty}{}_0P_\nu\,\Gamma\left(\frac{1+\nu}{2}\right)\frac{(1-e^{\pi i\nu})}{(n^{1/2})^{1+\nu}}$$
$$=\sum_{\nu=0}^{\infty}\frac{\Gamma(\nu+\frac{1}{2})\,{}_0P_{2\nu}}{n^{\nu+1/2}},$$

where ${}_0P_\nu$ is the coefficient of z^ν in the expansion in powers of z of the function

$$\left(\frac{1-\cos z}{z^2/2}\right)^{-(1+\nu)/2} e^{-z^3}.$$

Similarly

$$\int_{(2r-1)\pi}^{(2r+1)\pi} e^{2n(\cos z-1)}\,e^{-z^3} \sim e^{-(2r\pi)^3}\sum_{\nu=0}^{\infty}\frac{\Gamma(\nu+\frac{1}{2})\,{}_rP_{2\nu}}{n^{\nu+1/2}}$$

where ${}_rP_\nu$ is the coefficient of t^ν in the expansion in powers of t of the function

$$\left(\frac{1-\cos t}{t^2/2}\right)^{-(1+\nu)/2}\exp-[3(2r\pi)^2t+3(2r\pi)t^2+t^3].$$

Thus

$$\int_{-\pi}^{(2r-1)\pi} e^{2n(\cos z-1)}\,e^{-z^3} \sim \sum_{\nu=0}^{k} e^{-(2r\pi)^3}\sum_{\nu=0}^{\infty}\frac{\Gamma(\nu+\frac{1}{2})\,{}_rP_{2\nu}}{n^{\nu+1/2}}\ :$$

and we could readily justify the passage to the limit and assert that

$$\int_{-\pi}^{\infty} e^{2n(\cos z-1)}\,e^{-z^3}\,dz \sim \sum_{\nu=0}^{\infty}\frac{\Gamma(\nu+\frac{1}{2})P_{2\nu}}{n^{\nu+1/2}}$$

where

$$P_{2\nu}={}_0P_{2\nu}+e^{-(2\pi)^3}\,{}_1P_{2\nu}+e^{-(4\pi)^3}\,{}_2P_{2\nu}+\cdots.$$

* Cp. figure 2.41.

Generalized Quaternion Algebras and the Theory of Numbers.

By Lois W. Griffiths.

1. *Introduction.* The unified treatment of arithmetics of certain generalized quaternion algebras presented in this paper yields important theorems on representation of integers in quaternary quadratic forms, on the number of representations, and on the solution of Diophantine equations. The methods are essentially those applied by Dickson * to four algebras, two of which are instances in this paper.

2. *Generalized Quaternion Algebras.* A generalized quaternion algebra over the field of rational numbers has a modulus 1 and consists of all quantities $r = \sigma + \xi i + \eta j + \zeta k$, in which the *coordinates* σ, ξ, η, ζ are rational numbers, while

$$i^2 = \alpha, \quad j^2 = \tau, \quad k^2 = -\alpha\tau,$$
$$ij = -ji = k, \quad -jk = kj = \tau i, \quad -ki = ik = \alpha j,$$

in which α and τ are rational. The quantity r has the conjugate

$$\bar{r} = \sigma - \xi i - \eta j - \zeta k,$$

and norm

$$Nr = \sigma^2 - \alpha\xi^2 - \tau\eta^2 + \alpha\tau\zeta^2. \tag{1}$$

Hence for any two quantities r and s

$$Nr = r\bar{r} = \bar{r}r = N\bar{r}, \quad N(rs) = NrNs. \tag{2}$$

3. *Representation of Integers.* If α and τ are non-zero integers, as always henceforth, then the representation of an integer μ in the quaternary quadratic form

$$\sigma^2 - \alpha\xi^2 - \tau\eta^2 + \alpha\tau\zeta^2 \tag{3}$$

is equivalent to the existence in the algebra of a quantity whose coordinates are integers and norm is μ. For the algebras with parameters $(\alpha, \tau) = (-1, -1), (-1, -3), (-1, 3), (-1, -7)$, such quantities are integral

* *American Journal of Mathematics*, Vol. 46 (1924), pp. 1-16; *Algebren und Ihre Zahlentheorie*, Orell Füssli, Zurich (1927), Chapter IX; *Algebras and Their Arithmetics*, Chapter X, University of Chicago Press.

elements of the algebra, and the discussion of the arithmetics of these algebras given by Dickson * includes complete results of the type indicated in § 1 not only for the forms (3) but for allied forms in which cross product terms appear. The present paper derives similar results for the fifteen algebras with parameters

(4) $(-1,-3), (-2,-3), (-1,-7), (2,-3), (4,-3), (2,5), (-2,5), (3,5), (-3,5), (-4,5), (3,-7), (2,-11), (4,-11), (-2,13), (-4,13),$

by considering a set S which has all the properties of a set of integral elements except that it is not necessarily maximal.

The fundamental set S consists of all quantities

$$(5) \quad q = (\sigma + \eta/2) + (\xi + \zeta/2)i + j\eta/2 + k\zeta/2 = \sigma + \xi i + \eta H + \zeta L,$$

in which $H = (1+j)/2$, $L = (i+k)/2$, and the *coefficients* σ, ξ, η, ζ are integers. Hence S is closed under addition and multiplication, and contains the modulus. Furthermore, since

$$(6) \qquad Nq = \sigma^2 - \alpha\xi^2 + \epsilon\eta^2 - \alpha\epsilon\zeta^2 + \sigma\eta - \alpha\xi\zeta, \quad \epsilon = (1-\tau)/4,$$

is an integer, q satisfies the quadratic equation $x^2 - (2\sigma + \eta)x + Nq = 0$ with integer coefficients and leading coefficient unity. Hence S is a set of integral quantities if it is maximal, as in the two instances $(-1, -3)$, $(-1, -7)$. Henceforth all quantities considered are in S. If p, q, g are quantities of S such that $g = pq$, then g has a *left divisor* (factor) p and a *right divisor* q. A quantity of S which divides the modulus is called a *unit*. If u is a unit, gu is a *right associate* of g. A quantity which is not a unit is *prime in* S if it is not the product of two quantities of S each of which is not a unit.

The representation of an integer μ in the quaternary quadratic form

$$(7) \qquad \sigma^2 - \alpha\xi^2 + \epsilon\eta^2 - \alpha\epsilon\zeta^2 + \sigma\eta - \alpha\xi\zeta, \quad \epsilon = (1-\tau)/4,$$

is equivalent to the existence in S of a quantity whose coefficients are integers and norm is μ. Hence by (5) the representation of μ in (3) is equivalent to the representation of μ in (7) with η and ζ both even, that is, to the existence in S of a quantity whose coordinates are integers and norm is μ.

THEOREM 1. *Every positive and negative integer is represented in* (7)

---

* *Algebren und Ihre Zahlentheorie*, pp. 162-197; *Algebras and Their Arithmetics*, § 91, § 105.

with (α, τ) *of* (4), *except that, if* α *and* τ *are both negative, no negative integer, and, if* $|\alpha| = 4$, *no integer* $4^a 2\mu$ $(a \geqq 0, \mu$ *odd*$)$ *is represented.*

It is easily verified that

$$(8) \qquad \begin{aligned} &H + \mathrm{L},\ -1 - i + 2H + L,\ H,\ H,\ H,\ H,\ H,\ i + H, \\ &1 + H + L,\ -1 + 2H + L,\ 1 + H,\ 1 + H \end{aligned}$$

are units of negative norm for the algebras (4) respectively. Hence -1 and, by (2) and (7), $|\alpha|^a$ $(a \geqq 0)$ are represented. By (5) and (6), with $|\alpha| = 4$, $Nq \equiv 0 \pmod 2$ implies $\sigma \equiv \eta \equiv 0 \pmod 2$ and hence $Nq \equiv 0 \pmod 4$. Therefore it remains to prove the theorem for primes not dividing α. This is done by lemmas 1, 2, 3, whose proof is an extension of that by Dickson * with $(\alpha, \tau) = (-1, -7)$.

LEMMA 1. *If* π *is a prime which does not divide* α, *there are integers* λ, μ, ω *such that* $1 + \mu + \epsilon\mu^2 - \alpha\lambda^2 = \pi\omega$, $0 \neq |\omega| < \pi$, $0 \leqq |\lambda|, |\mu| \leqq \pi/2$.

LEMMA 2. *If* π *is an integer different from* 0, 1, -1, *and if* ω, λ, μ *are integers such that* $1 + \mu + \epsilon\mu^2 - \alpha\lambda^2 = \pi\omega$, *then* $0 \neq |\omega| < |\pi|$ *if if* $0 \leqq |\lambda|, |\mu| \leqq |\pi|/2$ *with* π *odd and if* $-(|\pi| - 2)/2 \leqq \lambda, \mu \leqq |\pi|/2$ *with* π *even.*

The proof consists in solving the inequality $-\pi^2 < 1 + \mu + \epsilon\mu^2 - \alpha\lambda^2 < \pi^2$ for the pairs of integers α, ϵ, subject to these conditions on λ, μ.

LEMMA 3. *If* π *is a prime which does not divide* α, *there is a quantity* g *of* S *whose norm is* π.

By lemma 1 there are integers λ, μ, ω such that $1 + \mu + \epsilon\mu^2 - \alpha\lambda^2 = \pi\omega$, $0 \neq |\omega| < \pi$. Give g the notation $A + Bi + CH + DL$, where

$$(9) \qquad \begin{aligned} A &= \pi\sigma - \alpha\lambda\xi + \epsilon\mu\eta, & B &= \xi, \\ E &= \pi\zeta + \mu\xi - \lambda\eta, & C &= \eta. \end{aligned}$$

Hence by (6) it is sufficient to prove the existence of integers σ, ξ, η, ζ such that

$$(10) \qquad \begin{aligned} 4\pi = \ & 4\pi^2\sigma^2 - 4\alpha\pi\omega\xi^2 + 4\epsilon\pi\omega\eta^2 - 4\alpha\epsilon\pi^2\zeta^2 \\ & + 8\alpha\pi\lambda(-\sigma\xi + \epsilon\eta\zeta) + 4\pi(1 + 2\epsilon\mu)(\sigma\eta - \alpha\xi\zeta). \end{aligned}$$

Since $\alpha\omega\pi \neq 0$, multiply (10) by $-\alpha\omega/\pi$ and write

$$(11) \qquad \sigma' = \alpha\xi, \quad \xi' = -\sigma, \quad \eta' = \alpha\zeta, \quad \zeta' = -\eta,$$

---

* *American Journal of Mathematics*, Vol. 46 (1924), pp. 11-13; *Algebren und Ihre Zahlentheorie*, § 101.

to obtain

$$(12)\qquad -4\alpha\omega = 4\omega^2\sigma'^2 - 4\alpha\omega\pi\xi'^2 + 4\epsilon\omega\pi\eta'^2 - 4\alpha\epsilon\omega^2\zeta'^2 + 8\alpha\omega\lambda(-\sigma'\xi' + \epsilon\eta'\zeta') + 4\omega(1+2\epsilon\mu)(\sigma'\eta' - \alpha\xi'\zeta').$$

This, by comparison with (9) and (10), expresses the fact that

$$N(A' + B'i + C'H + D'L) = -\alpha\omega, \quad \text{where}$$

$$(13)\qquad A' = \omega\sigma' - \alpha\lambda\xi' + \epsilon\mu\eta', \qquad B' = \xi',$$
$$E' = \omega\zeta' + \mu\xi' - \lambda\eta', \qquad C' = \eta'.$$

Now if $|\omega| \neq 1$ the least residues κ, ν of λ, μ modulo ω determine integers γ, β, π' such that $\lambda = \kappa + \omega\gamma$, $\mu = \nu + \omega\beta$, $1 + \nu + \epsilon\nu^2 - \alpha\kappa^2 = \omega\pi'$; furthermore, by lemma 2, $0 \neq |\pi'| < |\omega| < \pi$. Hence (13) become

$$(14)\qquad A' = \omega\sigma_1 - \alpha\kappa\xi_1 + \epsilon\nu\eta_1, \qquad B' = \xi_1,$$
$$E' = \omega\zeta_1 + \nu\xi_1 - \kappa\eta_1, \qquad C' = \eta_1,$$

where

$$(15)\qquad \sigma_1 = \sigma' - \alpha\gamma\xi' + \epsilon\beta\eta', \qquad \xi_1 = \xi',$$
$$\zeta_1 = \zeta' + \beta\xi' - \gamma\eta', \qquad \eta_1 = \eta',$$

so that (12) becomes

$$(16)\qquad -4\alpha\omega = 4\omega^2\sigma_1^2 - 4\alpha\omega\pi'\xi_1^2 + 4\epsilon\omega\pi'\eta_1^2 - 4\alpha\epsilon\omega^2\zeta_1^2 + 8\alpha\omega\kappa(-\sigma_1\xi_1 + \epsilon\eta_1\zeta_1) + 4\omega(1+2\epsilon\nu)(\sigma_1\eta_1 - \alpha\xi_1\zeta_1).$$

Substitute for $\sigma_1, \cdots, \zeta_1$ from (15) and (11), multiply by $-\pi'/\alpha\omega$, and write

$$(17)\qquad X = \xi + \gamma\sigma + \epsilon\beta\zeta, \qquad Y = \eta + \beta\sigma + \alpha\gamma\zeta$$

to obtain

$$(18)\qquad 4\pi' = 4\pi'^2\sigma^2 - 4\alpha\pi'\omega X^2 + 4\epsilon\pi'\omega Y^2 - 4\alpha\epsilon\pi'^2\zeta^2 + 8\alpha\pi'\kappa(-\sigma X + \epsilon Y\zeta) + 4\pi'(1+2\epsilon\nu)(\sigma Y - \alpha X\zeta).$$

Hence by (17) the solution of (18) in integers is equivalent to that of (10) in integers.

If $|\pi'| \neq 1 \neq |\omega|$, by the process applied to (12) to obtain (16), (18) yields an equation in π', ω', λ', μ' such that $0 \neq |\omega'| < |\pi'| < |\omega| < \pi$, which is precisely similar to (10) in π, ω, λ, μ, whose solution is equivalent to that of (10). If $|\omega| = 1$, $\gamma = \omega\lambda$, $\beta = \omega\mu$ implies $\kappa = \nu = 0$, $|\pi'| = |\omega| = 1$. If $\pi' = 1$, (18) has the solution $\sigma = 1$, $X = Y = \zeta = 0$; if $\pi' = -1$, -4π is to be used on the left of (10) initially and one of (8) at this point. Hence the proof by descent is complete.

Thus finally theorem 1 is proved.

LEMMA 4. *Except with* $(-1, -7)$, *there are in* S *three units* $u_1 = 1$, u_2, u_3 *of positive norm such that for every quantity* g *in* S *at least one of* $u_h g u_f$ $(h, f = 1, 2, 3)$ *has integral coordinates. If* $Ng \equiv \alpha \pmod 2$, *at least one of* gu_h $(h = 1, 2, 3)$ *has integral coordinates.*

Give the notations $g = \sigma + \xi i + \eta H + \zeta L$, $u = \kappa + \lambda i + \mu H + \nu L$. Then $ug = A + Bi + CH + DL$, where

$$(19)\quad \begin{aligned} A &= \kappa\sigma + \alpha(\lambda + \nu)\xi - \epsilon\mu\eta + \alpha\epsilon\nu\zeta, & C &= \mu\sigma - \alpha\nu\xi + (\kappa + \mu)\eta + \alpha\lambda\zeta, \\ B &= \lambda\sigma + (\kappa + \mu)\xi - \epsilon\nu\eta + \epsilon\mu\zeta, & D &= \nu\sigma - \mu\xi + (\lambda + \nu)\eta + \kappa\zeta, \end{aligned}$$

and similarly for gu. Then it is verified directly with $\sigma, \xi, \eta, \zeta \equiv 0, 1 \pmod 2$ in all ways, that the units $-1 + H$, H satisfy the requirements with the algebras $(-1, -3)$ $(2, -3)$, $(-2, -3)$, $(4, -3)$, that is, in abbreviated notation, with the algebras $(-1, \pm 2, 4; -3)$. Similarly $1+H$, $2-H$ with $(\pm 2, \pm 3, -4; 5)$; $-1 + i + H$, $i + H$ with $(2, -11)$, $(-4, 13)$; $2 + 3i + 3H$, $-5 + 3i + 3H$ with $(3, -7)$, $(4, -11)$, $(-2, 13)$. On the other hand, with $(-1, -7)$ the units are $1, -1, i, -i$, none of which satisfies for $g = H$.

THEOREM 2. *Every positive and negative integer represented in the form* (7) *of theorem* 1 *is represented in the form* (3), *except some integers not divisible by* 4 *with* $(\alpha, \tau) = (-1, -7)$.

This follows from (2) and lemma 4. Clearly 3 is not represented in the form (3) with $(-1, -7)$; whereas for multiples of 4 with $(-1, -7)$, the representation of μ in (7) is equivalent to that of 4μ in (3).

4. *Solution of Diophantine Equations.*

LEMMA 5. *A quantity* g *of* S *whose norm is divisible by a prime* π, $\neq 2$ *with* $|\alpha| = 4$, *has a right* (*left*) *divisor of norm* π.

By theorem 1 it suffices to consider $g \not\equiv 0 \pmod \pi$. Give the notation $g = \kappa + \lambda i + \mu H + \nu L$. If π divides α, $|\alpha| \neq 1, 4$. Then $Ng \equiv 0 \pmod \alpha$ implies $\kappa \equiv 0 \equiv \mu \equiv gi \pmod \alpha$, and since $Ni = -\alpha$, there is a quantity q such that $g = q\bar{\imath}$. Hence, if u is a unit whose norm is of the same sign as Ni, ui satisfies as a right divisor. If $\alpha \not\equiv 0 \pmod \pi$, the method of proof is that by Dickson (*ibid.*) with $(-1, -7)$. The three cases have hypo-

theses respectively $\kappa^2 - \alpha\epsilon\nu^2 \not\equiv 0 \pmod{\pi}$, $\epsilon\mu^2 - \alpha\lambda^2 \not\equiv 0 \pmod{\pi}$, $\kappa^2 - \alpha\epsilon\nu^2 \equiv 0 \equiv \epsilon\mu^2 - \alpha\lambda^2 \pmod{\pi}$. Except with $(-2, -3)$, $|\alpha| = 4$, there is a contradiction which makes the third case impossible as in Dickson's proof; otherwise, in the third case, $-1 + 2H$ satisfies as a divisor. This completes the proof of the lemma, since a right divisor p of $\bar{g}$ determines a left divisor $\bar{p}$ of g, and $Np = N\bar{p}$.

LEMMA 6. *With $|\alpha| = 4$, there are in S no quantities of norm 2 or -2 and there are quantities of even norm which have neither a left nor a right divisor of norm 4 or -4.*

The first statement is a corollary of theorem 1. It is verified by (19) and the conditions that $q = \kappa + \lambda i + \mu H + \nu L$ be such that $Nq = \pm 4$, $qg \equiv 0 \pmod 4$ or $gq \equiv 0 \pmod 4$, that $g = 4 - i + 2H + L$ is a quantity satisfying the second statement.

Lemmas 5 and 6 imply lemma 7 and theorem 3 on the solution of Diophantine equations.

LEMMA 7. *With $|\alpha| \neq 4$, if g is in S and $Ng = \delta_1 \cdots \delta_n$ there are quantities $d_1, \cdots, d_n$ in S such that $g = d_1 \cdots d_n$ and $\delta_h = Nd_h$. With $|\alpha| = 4$, a similar statement holds if not more than one of $\delta_1, \cdots, \delta_n$ is divisible by 2.*

THEOREM 3. *All integral solutions $\sigma, \xi, \eta, \zeta, \delta_1, \cdots, \delta_n$ of*

$$\sigma^2 - \alpha\xi^2 + \epsilon\eta^2 - \alpha\epsilon\zeta^2 + \sigma\eta - \alpha\xi\zeta = \delta_1 \cdots \delta_n, \quad \epsilon = (1-\tau)/4, \tag{20}$$

with (α, τ) of (4) and $|\alpha| \neq 4$, are obtained from $\sigma + \xi i + \eta H + \zeta L = d_1 \cdots d_n$, $\delta_h = Nd_h$, where $d_1, \cdots, d_n$ range independently over S. With $|\alpha| = 4$, precisely all integral solutions in which at most one of $\delta_1, \cdots, \delta_n$ is even, and that divisible by an even power of 4, are so obtained with at most one of $d_1, \cdots, d_n$ of even norm.

It is easily seen by lemma 4 that, with α odd and $(\alpha, \tau) \neq (-1, -7)$, if g has integral coordinates and π is an odd prime dividing Ng then there are quantities p, q in S, each having integral coordinates, such that $g = pq$ and $Np = \pi$. Hence there follows a lemma analogous to lemma 7, and

THEOREM 4. *All integral solutions, in which at most one of $\delta_1, \cdots, \delta_n$ is even, of*

$$\sigma^2 - \alpha\xi^2 - \tau\eta^2 + \alpha\tau\zeta^2 = \delta_1 \cdots \delta_n, \tag{21}$$

with $(\alpha, \tau) = (-1, -3), (3, 5), (-3, 5), (3, -7)$, *are obtained from* $\sigma + \xi i + \eta H + \zeta L = d_1 \cdots d_n$, $\delta_h = N d_h$, *where* $d_1, \cdots, d_n$ *range independently over* S *but are such that each has integral coordinates and at most one is of even norm.*

The remaining lemmas of § 4 add the uniqueness feature to the factorization of quantities in S. This is essential to the enumerations of § 5.

LEMMA 8. *If* $|\alpha| \neq 4$ *there are in* S *quantities* p_h ($h = 1$ *if* α *is even,* $h = 1, 2, 3$ *if* α *is odd*) *such that* $|Np_h| = 2$ *and that, if* g *is a quantity of* S *having* $Ng = 2^a\mu$ ($a > 0$, μ *odd*), *then there are unique quantities* m, q *in* S *such that* $g = mq2^b$, *where* 2^b *is the highest power of* 2 *dividing* g, $|Nm| = |\mu|$, $|Nq| = 2^{a-2b}$, *and* q *is a product of factors* p_h. *Furthermore* q *is a product of factors* p_h *in only one way.*

The proof for α even is in that of lemma 5. For α odd, by (6) and (19), it is verified that the following quantities $\bar{p}_h$ are of norm 2 or -2 and such that for any g one and only one of $\bar{p}_h g$ is divisible by 2: with $(-1, 3; -7)$ $H, -1 + H, 1 + i$; with $(-1, -3)$, $(3, 5)$ $1 + i, H + L, -1 - i + H + L$; *with* $(-3, 5)$ $i + H, -1 + i + H, i + H + L$. Then $g = p_h q$ as in lemma 5.

LEMMA 9. *If* π *is a prime and* m, p, q *are quantities of* S *such that* $Np = \pi$ *and* π *divides* mpq, *then* π *divides* mp *or* pq.

If α is divisible by π and $|\alpha| = 4$, the lemma is vacuously true. If α is divisible by π and $|\alpha| \neq 4$, then $\pi = |\alpha|$ and $mpq = \alpha r$. Hence α divides Nm or Nq, and by the proof of lemma 5 α divides mp or pq. The proof when α is not divisible by π follows that of Dickson (*ibid.*) with lemma 8 for the case $\pi = 2$.

LEMMA 10. *If* π *is a prime,* $\neq 2$ *with* $|\alpha| = 4$, *dividing* Ng *but not* g, *then* g *and* π *have a common left* (*right*) *divisor of norm* π, *determined uniquely up to a unit right* (*left*) *factor.*

This follows for $\pi = 2$ by lemma 8. For $\pi > 2$, by lemma 5 there are quantities p, q such that $g = pq$ and $Np = \pi$, and by lemma 9, if P is any common divisor of norm π, $0 \equiv \bar{P}g \equiv \bar{P}pq \pmod{\pi}$, π divides $\bar{P}p$, and P is a right associate of P.

A quantity is *proper* when and only when the greatest common divisor of its coefficients is unity. Hence by lemma 10 the factorizations of lemma 7

and preceding theorem 4 are, for the prime factors of Ng in fixed but arbitrary order, unique up to unit factors if g is proper. Hence finally, using lemma 10 and the method of Dickson, any two quantities g and q of S have a greatest common left (right) divisor determined uniquely up to a unit right (left) factor with $|\alpha| \neq 4$, and the same statement holds with $|\alpha| = 4$ if not both Ng and Nq are divisible by 2. For the algebra $(-1, -3)$ Dickson * established the existence of this greatest common divisor directly by the process of division with remainders of decreasing norms as in the theory of ordinary integers, and then proved the foregoing theorems on factorization of integral quantities, representation of integers and solution of Diophantine equations.

5. *Number of Representations.* If ν is any integer not excepted in lemma 16, there will be found finite numbers, $F(\nu)$ and $G(\nu)$ respectively, of sets of all and all proper quantities in S such that every quantity in S of norm ν or $-\nu$ is in one and only one set, and such that a given set consists of the right associates of any one of its members. Then the numbers of fundamental representations in the form (7), from which all and all proper representations are derived by unimodular transformations, are determined as $F(\nu)$ and $G(\nu)$.

If ν is prime to α and τ, $F(\nu)$ and $G(\nu)$ are found by the method of Dickson.†

LEMMA 11. *If π is a prime which divides neither α nor τ, $F(\pi) = \pi + 1$.*

For $\pi = 2$ this is lemma 8. For $\pi > 2$ the proof is Dickson's, with $\rho = \kappa\kappa_1 - \alpha\lambda\lambda_1 - \tau\mu\mu_1 + \alpha\tau\nu\nu_1$ here.

It is easily verified that if π is a rational prime and $Np = \pi$ then p is a prime quantity, and conversely. Hence lemma 10 implies lemma 12 and they in turn lemmas 13 and 14.

LEMMA 12. *A product of prime quantities is proper if and only if no factor is a right associate of the conjugate of its predecessor.*

LEMMA 13. *If μ is an integer prime to α and to τ and $\pi^a\kappa^b \cdots$ its prime factorization, then $G(\mu)$ equals $Q(\mu) = \mu(1 + 1/\pi)(1 + 1/\kappa) \cdots$.*

LEMMA 14. *If μ is an integer prime to α and to τ, $F(\mu)$ is the sum $f(\mu)$ of all the positive divisors of μ.*

---

* ***Algebras and Their Arithmetics***, p. 150; ***Algebren und Ihre Zahlentheorie***, p. 163.
† ***Algebren und Ihre Zahlentheorie***, pp. 175-179.

Lemmas 13 and 14 are extended to all integers, when possible, by enumerations for α and τ similar to those in lemmas 8 and 11.

LEMMA 15. *There are in S quantities p_h $[h = 0, 1, \cdots, 2|\tau|$ with $(-2, -3)$, $|\alpha| = 4$, $h = 0$ otherwise] such that $|Np_h| = |\tau|$ and that, if g is a quantity of S having $Ng = \tau^a\mu$ $[a > 0$, $\mu \not\equiv 0 \pmod{\tau}]$, then there are in S unique quantities m, q, s such that $g = \tau^b qm$, τ^b is the highest power of τ dividing g, $|Nm| = |\mu|$, $|Nq| = |\tau^{a-2b}|$, and q is a power of p_0 if $h = 0$, otherwise $q = p_0^c s$, $0 \leqq c \leqq a - 2b$, and s is a product of the p_h with $h = 1, \cdots, 2|\tau|$. Furthermore s is a product of these p_h in only one way.*

Give the notation $g = \sigma + \xi i + \eta H + \xi L$. Then by (6)

$$0 \equiv Ng \equiv 4Ng \equiv (2\sigma + \eta)^2 - \alpha(2\xi + \zeta)^2 \pmod{\tau}. \tag{22}$$

Hence, except with $(-2, -3)$, $|\alpha| = 4$, $2\sigma + \eta \equiv 0 \equiv 2\xi + \zeta \equiv (-1 + 2H)g \pmod{\tau}$, so that $p_0 = 1 - 2H$, of norm $-\tau$, satisfies for every g. The proof with $(-2, -3)$, $|\alpha| = 4$ is long. Thus, by (19), for σ, ξ, η, $\zeta \equiv 0, 1, \cdots, |\tau| - 1 \pmod{\tau}$ in all ways consistent with (22), $\bar{p}_0 g \equiv 0 \pmod{\tau}$ and hence p_0 is a left divisor of g if and only if $2\sigma + \eta \equiv 0 \pmod{\tau}$, and otherwise $\bar{p}_h$ as follows are satisfactory (upper signs taken together):

with $(-2, 4; -3)$ $\quad H \mp L, -1 \pm i, -1 \pm i + H \mp L;$

with $(-4, 5)$ $\quad -3 - H \pm 2L,\ 7 \pm 2i - 5H,\ \pm i - 3H \pm L,$
$-1 \mp 2i - H \mp 3L,\ -2 \mp 2i + H \mp 3L;$

with $(4, -11)$ $\quad 5 \pm 3i,\ \pm 3 - 2i + L,\ -4 \mp i + H,\ -3 \mp 2i + 3H,$
$4 + H \pm L,\ 2 \pm i + H \pm L,\ -3 \mp 2i + H \pm L,$
$-5 \mp i + H \pm L,\ 2i \pm 3H,\ 3 \pm i + H,$
$\mp 3 + i + L;$

with $(-4, 13)$ $\quad 3 \mp i,\ -5 \mp i + 3H,\ 2i \mp H,\ 1 \mp i + H \pm L,$
$3 \pm i + H,\ \mp 7 + 3i + 3L,\ -1 \pm 2i + 3H,$
$-2 \mp 2i + 3H,\ \pm 7 - 6i + 3L,\ -4 \mp i + H,$
$-2 + H \pm L,\ -1 \mp 2i + H,\ 2 \pm i + 3H.$

Furthermore only one of p_h $(h = 1, \cdots, 2|\tau|)$ is a left divisor unless $2\sigma + \eta \equiv 0 \pmod{\tau}$.

LEMMA 16. *If* $\nu = \alpha^a \tau^b \mu$ ($a = 0$ *with* $|\alpha| = 4$ *but otherwise* $a \geqq 0$, $b \geqq 0$, μ *relatively prime to* α *and to* τ), $F(\nu) = \beta f(\mu)$, *where* $f(\mu)$ *is the sum of all the positive divisors of* μ, $\beta = 1$ *except with* $(-2, -3)$, $|\alpha| = 4$, *and otherwise* $\beta = 1 + 2|\tau|(|\tau|^b - 1)/(|\tau| - 1)$.

By lemmas 5 and 14 it is sufficient to prove $F(\nu/\mu) = \beta$. If $a > 0$ with $|\alpha| = 4$ this would not be sufficient, by lemma 6. If $|\alpha| \neq 4$, α is a prime and, for every positive integer a, $F(\alpha^a) = 1$ by the proof of lemma 5. By lemma 15 $F(\tau^b) = 1$ except with $(-2, -3)$, $|\alpha| = 4$. With $(-2, -3)$, $|\alpha| = 4$ a product $p_h p_f$ has p_0 as a left divisor if and only if $h = 0$ or $f = 0$ or h and f are not both from the set $1, \cdots, |\tau|$ (here the quantities with upper signs have subscripts $1, \cdots, |\tau|$). Hence for any positive integer b each of the $F(\tau^b)$ sets contains one and only one of the quantities: all products of b factors with subscripts from $1, \cdots, |\tau|$, all products of b factors with subscripts from $|\tau| + 1, \cdots, 2|\tau|$, and all products $p_0 q$ where q is in turn one quantity from each of the $F(\tau^{b-1})$ sets. Hence by induction $F(\tau^b) = 1 + 2|\tau|(|\tau|^b - 1)/(|\tau| - 1)$. Finally $F(\nu/\mu) = F(\alpha^a)F(\tau^b) = \beta$, as stated.

THEOREM 5. *All representations of* ν, *positive with* $(-1, -3)$, $(-2, -3)$, $(-1, -7)$, *in the form* (7) *are obtained from* $F(\nu) = \beta f(\mu)$ *fundamental representations by all transformations of determinant*

$$(23) \qquad \begin{vmatrix} \kappa' & \alpha\lambda' & -\epsilon\mu' & \alpha(\lambda' + \epsilon\nu') \\ \lambda' & \kappa' & \lambda' + \epsilon\nu' & -\epsilon\mu' \\ \mu' & \alpha\nu' & \kappa' + \mu' & -\alpha\lambda' \\ \nu' & \mu' & -\lambda' & \kappa' + \mu' \end{vmatrix} = \kappa'^2 - \alpha\lambda'^2 + \epsilon\mu'^2 - \alpha\epsilon\nu'^2 + \kappa'\mu' - \alpha\lambda'\nu' = 1.$$

The notations are those of lemma 16. Each of the $F(\nu)$ fundamental sets contains a quantity of norm ν by (8), and every quantity qv, where $Nq = -\nu$ and $Nv = -1$, is in one of these fundamental sets. Hence the quantities of norm ν are right associates of $\beta f(\mu)$ quantities. Write $g = \sigma + \xi i + \eta H + \zeta L$, $u = \kappa' + \lambda' i + \mu' H + \nu' L$. Then by (19) the coefficients of gu are linear homogeneous functions of $\sigma, \cdots, \zeta$ with determinant (23). Hence by (2) the theorem follows.

THEOREM 6. *Except with* $(-1, -7)$, *if* $\nu \equiv \alpha \pmod 2$, *and if* $\nu > 0$ *with* $(-1, -3)$, $(-2, -3)$, *all representations of* ν *in the form* (3) *are obtained from* $\beta f(\mu)$ *fundamental representations by transformations of determinant* (23), *but some quadruples so obtained may not represent* ν *in* (3).

This follows from theorem 5 by lemma 4.

LEMMA 17. *If* $\nu = \alpha^a \tau^b \mu$ $(a = 0$ *with* $|\alpha| = 4$ *but otherwise* $a \geqq 0$, $b \geqq 0$, μ *relatively prime to* α *and to* $\tau)$, $G(\nu) = \rho Q(\mu)$, *where* $\rho = 0$ *if* $a > 1$ *or* $b > 1$ *and otherwise* $\rho = 1$ *with* $(\alpha, \tau) \neq (-2, -3)$, $|\alpha| = 4$; *where* $\rho = 0$ *if* $a > 1$ *and otherwise* $\rho = 1$ *if* $b = 0$, $\rho = 2|\tau| + 1$ *if* $b = 1$, $\rho = 2|\tau|^{b-1}(|\tau| + 1)$ *if* $b > 1$ *with* $(-2, -3)$, $|\alpha| = 4$.

By lemmas 13 and 5 it is sufficient to prove $G(\nu/\mu) = G(\alpha^a) G(\tau^b) = \rho$. With $|\alpha| \neq 4$, $(\alpha, \tau) \neq (-2, -3)$, by the proof of lemma 16, $G(\alpha^a) G(\tau^b) = 0$ if $a > 1$ or $b > 1$ and otherwise $G(\alpha^a \tau^b) = 1$. With $(-2, -3)$ $\rho = 0$ if $a > 1$. If $a = 0$ or 1 with $(-2, -3)$ and if $a = 0$ with $|\alpha| = 4$, $\rho = 1$ if $b = 0$ and $\rho = 2|\tau| + 1$ if $b = 1$. Finally if $b > 1$ each of the $F(\tau^b) - G(\tau^b)$ sets of quantities of norms $\pm \tau^b$ which are not proper contains one and only one of the quantities $p_0 p_0 q$, where q is in turn one quantity from each of the $F(\tau^{b-2})$ fundamental sets. Hence $G(\tau^b) = F(\tau^b) - F(\tau^{b-2}) = 2|\tau|^{b-1}(|\tau| + 1)$.

THEOREM 7. *All proper representations of* ν, *positive with* $(-1, -3)$, $(-2, -3)$, $(-1, -7)$, *in the form* (7) *are obtained from* $G(\nu) = \rho Q(\mu)$ *fundamental representations by transformations of determinant* (23).

The notations are of lemma 17. This theorem is not extended to the form (3) since there are quantities in S for which the greatest common divisor of the coefficients is different from unity while that of the coordinates is unity.

With $(-1, -3)$, $(-2, -3)$, $(-1, -7)$ there are precisely 12, 6, 4 units respectively since the forms (7) and (3) are positive. Then theorems 5, 6, 7 are theorem 8 and the classic theorem 9.

THEOREM 8. *All* {*all proper*} *representations of* $|\alpha^a \tau^b N|$ $(a, b \geqq 0$, N *prime to* α *and* $\tau)$ *in* (7) *with* $(\alpha, \tau) = (-1, -3)$, $(-2, -3)$, $(-1, -7)$, *are in number* $12f(N)$, $6[1 + 3(3^b - 1)]f(N)$, $4f(N)$ $\{12\rho Q(N), 6\rho Q(N), 4\rho Q(N)\}$.

THEOREM 9. *All representations of* $2^a 3^b N$ $(a > 0$, $b \geqq 0$, N positive *and prime to* 6) *in the form* $\sigma^2 + 2\xi^2 + 3\eta^2 + 6\zeta^2$ *are in number* 2 *or* 6 *times* $1 + 3(3^b - 1)$ *times the sum of all the positive divisors of* N *according as* $a = 1$ *or* $a > 1$.

This result was obtained otherwise by Liouville.* Similar theorems for the form (3) with $(\alpha, \tau) = (-1, -3), (-1, -7)$ have been obtained by the present method by Dickson.†

The methods of this paper have been applied to the algebras $(2, -15)$, $(3, -15)$, $(-3, 17)$, but the results are not included here since certain exceptions would recur in the theorems. The results of § 3 have been established for four infinite systems of algebras by a modification of that method which does not, however, permit of application to obtain the results of § 4, 5. These algebras seem to be the only ones to which the method of § 3 is applicable.

* Dickson, *History of the Theory of Numbers*, Vol. 3, p. 229.
† *Algebren und Ihre Zahlentheorie*, Chapter IX.